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THE TRANSITIONAL DYNAMICS OF DEBT: PRODUCTIVITY SHOCKS, FISCAL POLICY AND ECONOMIC GROWTH

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ABSTRACT

We propose a two-sector endogenous growth model with both public and private production. The government sector produces an output that enters both the utility function and the production function, financed by both taxes and borrowing, such that public expenditure and public finance are independently determined. We impose a homogeneous stylized labour productivity supply shock to country specific simulations, coinciding with hypothetical year 2020, and consider their respective heterogeneous fiscal responses. The simulations depict the transitional dynamics in both developed and less developed economies to the simultaneous labour and policy shocks.

Using advanced software applications, explicitly suited to model continuous time mathematics, we simulate closed form solutions to continuous time general equilibria that are defined by a three-dimensional modified golden rule across consumption, capital and debt. The model represents a baseline for analysis of the short to medium term impacts of shocks, both exogenous or policy induced, within a richly defined dynamic general equilibrium. Our methodology extends the ability to analyze transitional dynamics in multidimensional models where hitherto analysis has been confined to the long run equilibria.

INTRODUCTION

We propose a two-sector endogenous growth model with both public and private production. The government sector trivially produces an output, financed by taxes and or borrowing, that enter both the utility and production functions. The publicly produced goods increase agent welfare directly through a pure public good within the utility function that is subject to congestion. In addition, publicly produced goods serve as an intermediate productive input within the private production function. Public expenditure and public finance are determined independently of one another where government deficits play an accommodating role within the general equilibrium.

Our framework facilitates analysis of complex fiscal policies on growth and welfare under the presumption of independence between public spending and public finance. For any given level of public expenditure, taxes pose a twofold problem for the government. In addition to impacting the transitional dynamics through its indirect impact on debt, distortionary taxes also impact the long run steady state. An effective tax rate set above or below the balanced budget rate will necessarily lower steady state consumption and by extension, the rest of the saddle path. The government must therefore simultaneously address the transitional consumption tradeoff resulting from debt decisions as well as the long-term consideration of sub-optimal taxation on the steady state. The tax distortion extends further with the inclusion of both direct and indirect taxation.

We create directly comparable simulated consumption paths for various countries using real data for initial state conditions in addition to informed coefficient values. We then consider a homogeneous labour productivity shock within each country at the hypothetical year 2020 that will necessarily rectify itself in time, at a diminishing rate, with or without government intervention. The governments in question can mitigate the shock with income support to the representative agents and or increased provision of unproductive public goods, either policy financed by public debt.

The model represents the dynamic tradeoff that occurs from government deficit spending shock within the general equilibrium. While the debt has no long run impact on the steady state as per Ricardian equivalence, the public debt has significant short run transitional impact as can be seen from the consumption paths.

We consider a highly stylized representation of the 2020 Covid-19 pandemic modelled as a temporary labour productivity shock. The supply shock in our model is analytically similar in form to Gori, et.al. (2022), except we assume a homogeneous labour supply shock.

The typical government response to the 2020 Covid-19 pandemic has been deficit financed income support ranging from relatively generous income support above 10 percent, as in Switzerland and the United States, to more moderate levels under 5 percent of GDP as in most less developed countries. While the outcomes of these policies on immediate and future welfare are largely still to be determined, anecdotal evidence suggests that income support, offered simultaneously as the pandemic unfolded, has been successful in mitigating the otherwise expected drop in consumption. Nevertheless, the impact of the unplanned public deficits on growth in the near and distant future is yet to be realized.

Our analytical framework demonstrates the dynamic effects of public deficits and debt when tax and spend decisions are determined independently of one another. We provide insights into the short- and medium-term impacts on economic growth and welfare of income support financed by public debt in response to a productivity shock. Our methodology quantifies the short run dynamic trade-off between consumption today versus tomorrow that results from the government borrowing from its future revenue stream to subsidize the private sector for one period. We comment on what the simulations predict in the short and medium terms for the countries in the sample in relation to the United States. We base our conclusions on numerical simulations conducted using the continuous time modeling software package, Altair Embed©, which is more common to engineering and research in fluid dynamics. Our modeling technique follows that of Barreto (2018). The model and the proposed analytical methodology represent the baseline for potential extensions into the transitional dynamics inherent to public policy, taxes and debt as well as those inherent to the consumer welfare function that includes public goods. As example of the model's applicability, a recent contribution by Alm and Barreto (2024) extends the consumer problem to include trust in government and its impact on tax compliance.

MODELING DISCUSSION

Our analytical approach, based on the Ramsey-Cass-Koopmans[1] (RCK) model expressed in continuous time as in Lucas (1988), defines the transitional saddle path of consumption to its steady state equilibrium level. Economic growth happens exclusively during the short run transition. In the long run, economic growth per-effective capita is zero. Debt is introduced as a third state dimension, and like consumption and capital, settle at long run steady state levels that are held together by endogenously determined saving. Although a sustainability constraint conceptually exists in the sense that we reject any fiscal policy that necessarily results in negative consumption along any point along its saddle path, since there is no analytical constraint over how much the government can borrow at any given time and subsequently pay back across the infinite horizon, we do not explicitly define a sustainability constraint on the debt.

[1] Ramsey (1928), Cass (1965), Koopmans (1965)

We propose a three-dimensional modified golden rule for steady states of per-effective capita consumption, capital and debt. Their growth is observed by their unique saddle paths. As such, we are primarily concerned with the transitions. We assume that all countries are in fact somewhere along their respective transitional paths with the long run intuitively as well as analytically in the distant future. Our simulations begin at a hypothetical time zero, representative of 1950 by initial state conditions, and we analyze the first one hundred and seventy simulated years, comparing the first sixty-five to real data from 1950 until 2015. We then impose a labour productivity shock at period seventy, representing the 2020 Covid-19 pandemic, and record the remaining eighty years of simulated effects with or without public debt financed income support. The models run for one thousand periods (years) to ensure sufficient time to converge to a steady state.

We presume that governments make spending decisions almost completely independent of whether the subsequent debt is sustainable in the long run or not. The policy independence of government finance versus expenditure, notwithstanding their effective interdependence within the general equilibrium, implies that debt plays an accommodating role. For any independently determined tax base, the government may raise or lower its relative expenditure rate above or below the balanced budget, independent of any concept of optimal expenditure. Analytically, public debt and deficits allow this to take place. Alternatively, for any given rate of expenditure, deficits and debt can accommodate almost any tax policy mix of income and consumption taxes, irrespective of whether the policy is growth, welfare, revenue or politically motivated.

Publicly produced goods are generally categorized in the growth literature as either productive or non-productive (Irmen and Kuehnel, 2009; Chatterjee and Ghosh, 2011; Escobar-Posada and Monteiro, 2015). We assume that productive public goods[2] are those that enter the production function while unproductive public goods are those that enter the consumer's utility function. As far as we know, the entire theoretical literature assumes an expenditure constraint over productive versus unproductive public goods such that there is a zero-sum gain in the distribution of public expenditure.[3] In other words, a dollar spent to produce a productive public good is a dollar less that is available for unproductive public goods. Public goods are thus assumed to be perfectly separable in their use. If roads are productive public goods and thereby enter the production function while schools are unproductive public goods and thereby enter the utility function, improvement of roads does not directly help consumers while improvement of schools does not directly help producers.

We explicitly consider public goods, subject to congestion, that simultaneously flow to both consumers via a utility function as well as to producers via the production function. By relaxing the public expenditure constraint, the effective demand for real public goods includes both the normal demand by private production and the non-exclusive demand by consumers. The supply of public goods flow into the private production where they are absorbed into final goods. Simultaneously, those same public goods, subject to congestion, flow through the consumer utility function. Consumer utility is accrued from per capita consumption and aggregate public goods. The distinction between per capita and aggregate allows the public goods their non-exclusive quality. Analytically, it implies the limit of the market influence of consumers on public goods is zero such that intuitively, consumer enjoyment of public goods does not diminish them.

Public goods within the utility function increases the optimal size of government. The omnipotent social planner increases utility at the expense of consumption to take advantage of the benefits of public goods. Global optimality implies a relatively larger government, which may or may not include deficit spending. This is distinct from the decentralized optimal government size given direct and indirect tax rates are set independently of public expenditure. As such, the optimal government size, when finance is exogenous, is smaller given the smaller revenue base.

Our analytical model captures policy independence between taxation and expenditure yet maintains their tractable general equilibrium interdependencies. We confirm the notion that public finance has both growth and distributional effects. We can compare optimal expenditure with a sub-optimal tax mix versus sub-optimal expenditure with an optimal tax mix or any other mix of expenditure, taxes and debt.

Consistent with much of the growth literature, we find support for Ricardian equivalence in that debt has no impact on the steady state. Nevertheless, we identify the non-monotonic impact of debt on the short and medium run transitional dynamics prior to the steady state. The social planner can intertemporally substitute consumption now for the future to raise growth rates today. Alternatively, the transition can drag out longer with necessarily higher interim rates of consumption at lower growth rates. The model demonstrates the stylized dynamic adjustment of greater debt and consumption today offset by lower consumption tomorrow.

The endogenous growth literature that considers public deficits generally focuses on the dynamic sustainability of debt. Following Greiner (2007, 2008), the intertemporal government constraint is defined by the primary surplus to GDP. Government debt is sustainable if the present value of the public debt converges asymptotically to zero (Greiner and Fincke, 2016, p. 7). Based on the AK model that necessarily defines equilibrium growth in the long run, the models consider the conditions under which public debt is dynamically sustainable with balanced growth in consumption.

[2] For ease of exposition, we often use 'public goods' to refer to publicly produced goods notwithstanding the obvious error in definition.

[3] Felice (2016) considers a productive public good that simultaneously flows to two different productions functions.

The role of taxation in the endogenous growth setting has been widely researched. Optimal income taxation in a dynamic environment generally depends on the manner the subsequent public spending enters the productive process. In general, income taxes inhibit growth when government spending is unproductive (Jones and Manuelli, 1990; Rebelo, 1991) and assist growth when spending is productive (Barro, 1990; Futagami, et.al., 1993). These papers, and many that follow, assume government expenditure are financed by tax revenues, most often within a balanced budget.

There are two issues inherent to that approach. The first concerns the continued debate pertaining to causality between taxation and expenditure. Although there may be some causality in either direction, it is not clear which way it goes. The tax-spend hypothesis, following Wagner (1976), Buchanon and Wagner (1977) and Friedman (1978), argues unidirectional causality from revenues to expenditure while the spend-tax hypothesis, following Barro (1979) and Peacock and Wiseman (1979), proposes the direction of causality is from expenditures to revenue. The fiscal synchronization hypothesis, following Musgrave (1966) and Metzer and Richard (1981), assumes bidirectionality, such that revenues and expenditures are simultaneously determined. The absence of causality, citing institutional independence amongst public agencies, follows Baghestani and McNown (1994). Our work falls into the last category. The second issue concerns whether or not public expenditure is in fact deficit financed. Recent evidence suggests that numerous countries, including six of the seven G7 nations, financed public expenditure by deficits at least from 1993 until 2012 (Kamiguchi and Tamai, 2012) resulting in high levels of debt to GDP as opposed to being funded by tax revenue.

THE MODEL – LUMP SUM TAX FINANCING

The model is intuitively simple; it is represented by a three-dimensional modified golden rule across consumption, capital and debt. It is nevertheless somewhat analytically cumbersome. Considering this latter aspect, we break down the analysis into a basic framework for public goods financed by lump sum taxes followed by two extensions according to the nature of government financing. The first extension assumes incomplete lump sum taxes and allows for public debt to cover any deficit or surplus. The goal is to analyze debt financing of public goods exclusive of any tax distortions. The second extension, the complete model, allows for a tax base of consumption and income taxes, with any deficit covered by government debt. We analytically define an effective tax rate that when set equal to the exogenously rate of government spending, optimizes growth and consumption.

On the demand side, the representative agent accrues utility through a Cobb-Douglas function of per capita final goods consumption and a public good. The elasticity on consumption is Φ. The coefficient of relative risk aversion and the economy wide discount rate are θ and ρ, respectively. The public good's non-exclusivity is captured in the utility function by explicitly combining an aggregate variable, common good M_t, [4] and consumption per capita, $\frac{C_t}{L_t}$. The representative agent maximizes lifetime welfare as follows.

$$
\begin{aligned}
Max\ W &= \int_{t=0}^{\infty} U\left(\frac{C_t}{L_t}, M_t\right) \cdot L_t e^{-\rho t} dt \\
&= A_0^{\phi(1-\theta)} L_0 \int_{t=0}^{\infty} \frac{\left(c_t^{\phi} \cdot M_t^{1-\phi}\right)^{1-\theta}}{1-\theta} \cdot e^{\left[n+\phi(1-\theta)\chi-\rho\right]t} dt
\end{aligned}
\tag{II.1}
$$

Lower case letters represent per-effective capita and upper case represent levels such, such that $c_t = \frac{C_t}{A_t L_t}$. Labour L_t and technology A_t grow at exogenously determined rates, n and χ, respectively.

$$
L_t = L_0 e^{nt} \quad \& \quad A_t = A_0 e^{\chi t} \tag{II.2}
$$

The agent faces a private resource constraint (PRC) where Y_t is income, C_t is consumption and T_t is taxes. Savings is represented by the change in capital, $\dot{K}_t$.

$$
Y_t = C_t + \dot{K}_t + T_t \tag{II.3}
$$

The common good is subject to congestion, $0 \le \gamma \le 1$, and is created trivially from government expenditure, G_t.

[4] A common good is generally defined as rival and non-excludable.

$$M_t = G_t^{\gamma}\left(\frac{G_t}{Y_t}\right)^{1-\gamma} \quad \textbf{(II.4)}$$

We assume an exogenous rate of public expenditure, σ , determines the size of the public sector, G_t. Taxes, T_t are levied lump sum. The government budget constraint is therefore expressed as equation (III.6). There is no debt because taxes are determined endogenously by expenditure.

$$G_t = \sigma Y_t \quad \textbf{(II.5)}$$

$$G_t = T_t \quad \textbf{(II.6)}$$

Most of the literature addressing taxation with exogenous labour supply and debt assume labour as the numeraire such that it plays no specific role. For example, Turnovsky (1996) assumes an AK model in production. Consequently, the "public" good in the utility function that interacts with consumption, which is per capita by assumption only, is itself aggregate, also by assumption only. What is subsequently lost is the non-exclusive nature of public goods which we represent by per capita consumption's interaction with the aggregate public good.

On the supply side, the public goods enter the production function as in Barro (1990). We assume that there is no congestion in the provision of public goods to production.

$$Y_t = K_t^{\alpha} G_t^{\beta}\left(A_t L_t\right)^{1-\alpha-\beta} \quad \textbf{(II.7)}$$

Note the explicit inclusion of labour in the production function that is subject to diminishing returns. Here lies the fundamental difference between the RCK framework, of which this is an extension, and the AK framework that defines most of the literature on debt and endogenous growth. Analytically, the AK model necessarily implies growth at the long run equilibrium. The RCK model defines steady state levels in the long run and demonstrates growth during the transition.

The model is summarized by the present value Hamiltonian which results in a modified Euler equation for the equilibrium growth rate of per-effective capita consumption, $\xi_t = \frac{\dot{c}_t}{c_t}$.

$$\mathbf{H} = U\left(c_t, M_t\right) + \lambda_t\left[\left(1-\sigma\right)y_t - c_t - \left(n+\chi\right)k_t\right] \quad \textbf{(II.8)}$$

$$\xi = \frac{1}{\Delta}\left[\left(1-\sigma\right)r + \Omega\frac{c_t}{k_t} + \Lambda\left(\chi+n\right) - \Phi\right] \quad \textbf{(II.9)}$$

For all intents and purposes, the Euler equation (II.9) remains true to form. The three defining components of the growth rate are the effective return on capital, consumption relative to capital stock[5] and a time invariant component. Note the lack of time subscript on growth, ξ. Although the growth rate changes over time, it is nevertheless not differentiated with respect to time wherever it may appear in subsequent formulae.

The time invariant components are readily dissected as follows.

$$\Delta = 1 - \phi\left(1-\theta\right) \quad \textbf{(II.10)}$$

$$\Omega = \frac{\alpha\gamma\left(1-\phi\right)}{\phi\left(1-\beta\right)} \quad \textbf{(II.11)}$$

[5] At the steady state, $\frac{c_\infty}{k_\infty} = \frac{y_\infty}{k_\infty}\left(1-\sigma\right) - \left(n+\chi\right)$. For all $t < \infty$, the difference between $\frac{c_t}{k_t}$ and $\frac{y_t}{k_t}$ is the growth rate of capital, $\frac{\dot{k}_t}{k_t}$.

$$\Phi = \rho + \chi\left[1-\phi\left(1-\theta\right)\right] \quad \textbf{(II.12)}$$

$$\Lambda = \left(1-\phi\right)\left(1-\theta\right)\gamma \quad \textbf{(II.13)}$$

Δ represents the effective coefficient of relative risk aversion. It defines proportional relationship between consumption growth and changes in its shadow price given the intertemporally substitutable public goods. Ω represents the proportional impact of consumption relative to capital on the shadow price of capital and consequently consumption growth Φ represents the effective discount rate, given the presence of public goods and technological growth. Λ defines the
proportional impact of congested public goods. Given any reasonable selection of coefficients,[6] public goods effectively temper the relative impact of consumption on welfare. In the simulations, = Λ 0.00075 suggests a very limited proportional impact of congested public goods on growth.

Finally, the saving rate may be defined as follows.

$$s = \frac{\left(\xi+\chi+n\right)\left(1-\sigma\right)\left(\dfrac{\alpha}{1-\beta}\right)}{\Delta\xi - \Lambda\left(\dfrac{\alpha}{1-\beta}\xi+\chi+n\right)+\Phi-\Omega\dfrac{c_t}{k_t}} \quad \textbf{(II.14)}$$

Although the saving rate plays no analytical role whatsoever in the solving the model, it nevertheless serves to check the validity of the model given its general equilibrium nature.[7] Note the limited, albeit positive impact of public goods on the saving rate, as defined by $\Lambda\left(\bullet\right)$.

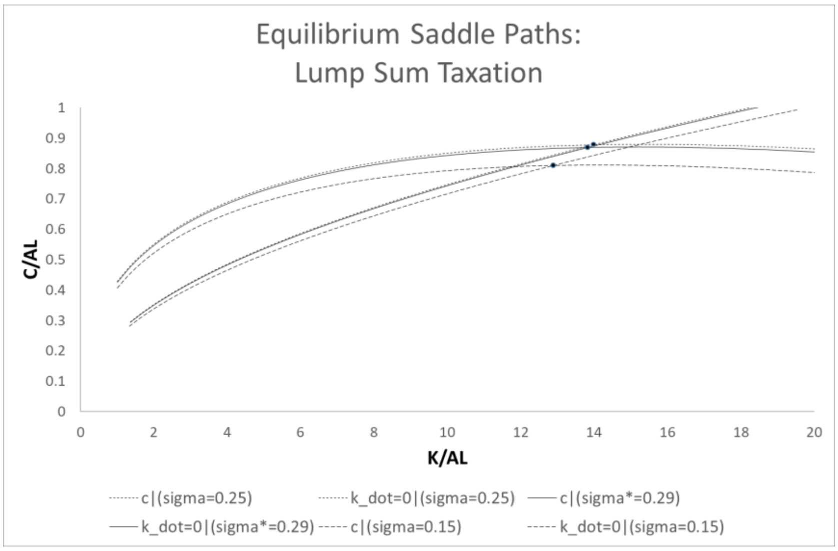

Figure 1

A_0=1, $K_{0(LHS)}$=1, $K_{0(RHS)}$=20 L_0=1, α =0.33, β =0.25, φ =0.90, γ =0.75, ρ =0.03, θ =0.99, χ =0.02, n =0.025

Figure 1 depicts the saddle paths of consumption for three distinct scenarios relating to the exogenous size of government. Each of the three saddle paths is comprised of two distinct simulations, one from the left of the steady state[8] and the other from the right of the steady state.[9]

The simulation determines the evolution of capital by $\dot{K} = \left(\chi+n+\xi\right)K_t$ given any set of initial conditions such as $A_0 = 1$, $K_0 = 1$ and $L_0 = 1$. The solution is a deterministic saddle path to a stable steady state. Given the one-sided

[6] All simulations share the following coefficient values, α=0.33, β=0.25, ϕ=0.90, γ=0.75, ρ=0.03, θ=0.99, χ=0.02, n=0.025 such that Δ=0.9910, Ω=0.0367, Λ=0.00075 and Φ=0.04982.

[7] The simulations solve for $C=Y-S$ where $S = \dot{K}$ which is procedurally distinct although analytically identical to $C=Y-S$ where $S=sY$.

[8] Initial conditions from the left-hand side of the steady state are A_0=1, K_0=1 and L_0=1.

[9] Initial conditions from the right-hand side of the steady state are A_0=1, K_0=20 and L_0=1

nature of the public sector, we can directly define optimal size of government by differentiating the present value Hamiltonian with respect to G_t to yield the following optimality condition.

$$\begin{aligned}\sigma^* = \frac{G_t^*}{Y_t} &= \frac{1-\phi}{\phi}\left(1-\beta+\gamma\beta\right)\frac{c_t}{y_t}+\beta \\ &= \frac{1-\phi}{\phi}\left(1-\beta+\gamma\beta\right)\varpi+\beta\end{aligned} \quad \textbf{(II.15)}$$

Although optimal government size is increasing with proportional consumption, the proportion itself $\frac{c_t}{y_t}=\varpi$ is time invariant such that the optimality condition is also static. Optimality is increasing in the congestion coefficient γ, while decreasing in elasticity of consumption φ. Increased congestion in the provision of public goods decreases the value of public goods and thereby lowers their optimal provision. At the same time, as the importance of public goods' relative contribution to overall utility decreases, so does the optimal size of government. The two effects together raise the optimal size of government above its elasticity in private production β. Optimal government size is increasing in the government production coefficient β, except for the extreme case of very high congestion, represented by a low value for γ, combined with very high dependence on public goods, represented by a low value for φ. It is worth noting that the model analytically defines the wedge between optimal provision of public goods from the producer's perspective which necessarily implies σ = β versus the needs for public goods by consumers who are willing to sacrifice some consumption, c_t, for more common good, M_t, to achieve higher welfare.

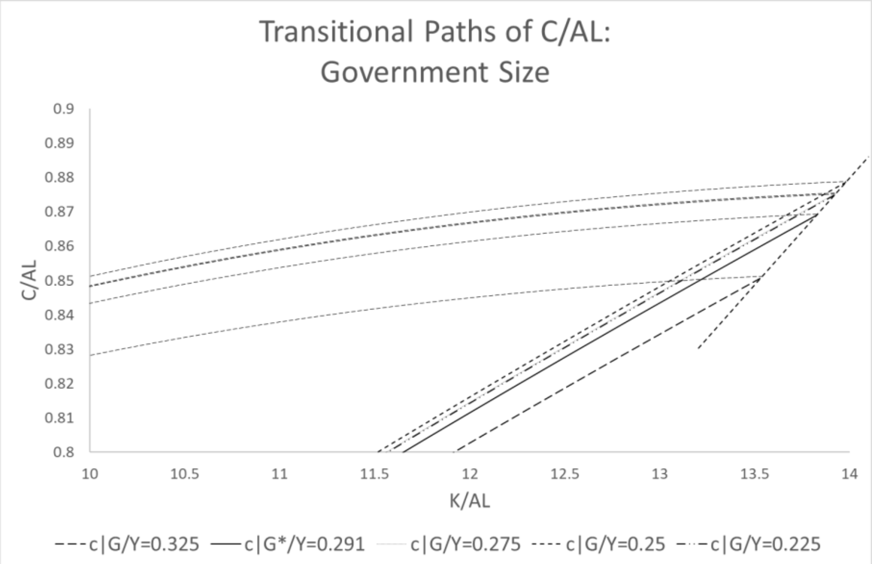

Figure 2

A_0=1, K_0=1, L_0=1, α =0.33, β =0.25, φ =0.90, γ =0.75, ρ =0.03, θ =0.99, χ =0.02, n =0.025

Figure 2 depicts the transition from the left of the steady state to modified golden rule where the $\dot{k}_t = 0$ locus meets the $\dot{c}_t = 0$ locus at the steady states of per-effective capita consumption and capital. The highest steady state consumption, and by extension the highest average growth rate throughout the transition, occurs when the proportion GDP devoted to government spending equals the elasticity of productive government in the production function, $\sigma = \beta = 0.25$. Notice the almost symmetric impact on consumption and growth of changes in government size by comparing either $\sigma = 0.225$ or $\sigma = 0.275$ to $\sigma = \beta = 0.25$. Also observe that when $\sigma = \beta = 0.25$, the agent has effectively further to go in the same space of time and thereby adopts a higher average growth rate.

The impact of public goods can be seen by comparing the optimal government size as determined by maximum consumption and growth, $\sigma = \beta = 0.25$, versus optimal government size as determined by utility, $\sigma^* = 0.291$. At the steady state limit, while the agent under optimal government size is operating at a marginally higher level of utility, his growth in welfare is independent of government size. Government size affects marginal utility monotonically during the transition but ceases to do so at the steady state. In the long run, optimality in relative government size affects only the level of welfare not its growth.

The size of government spending does not appreciably impact growth. Herein public spending has no transitional impact on either marginal consumption or marginal utility. Optimality of government allows the underlying economy to operate at higher levels of consumption and or utility per capita. The result implies that government spending devoted to both productive and unproductive public goods does not affect growth in the short run, per se.

THE MODEL – PUBLIC DEBT FINANCING

Suppose again that relative government spending, □, is policy driven and independent of revenue. Lump sum taxes that meet expenditure needs, as we assumed above, is tantamount to imposition of a balanced budget. Suppose instead, taxes are lump sum but incomplete. Consider independence between government expenditure and tax revenue. Assume that lump sum taxes are set arbitrarily at some time and grow at some exogenous rate different to the exogenous rate of government expenditure.

The changes to the model are evident in the changes to the consumer's and the government's budget constraints. The government budget constraint (GBC), defined as the aggregate resource constraint (ARC) minus the private resource constraint (PRC), is the expression for the evolution of bonds.

$$\dot{B}_t = \sigma Y_t - T_t + rB_t \ \text{: GBC} \tag{III.1}$$

$$\dot{K}_t = Y_t - C_t - G_t \ \text{: ARC} \tag{III.2}$$

$$\dot{K}_t + \dot{B}_t = Y_t + rB_t - C_t - T_t \ \textbf{:} \text{ PRC} \tag{III.3}$$

Let T_t be some arbitrarily determined level of taxation and B_t is the current stock of bonds. Notice the government undertakes exogenous expenditure σY_t and receives a lump sum T_t but must pay interest on its debt, rB_t. The government deficit or surplus is the difference made up by consumers whose budget constraint in per-effective capita terms now includes new bond purchases on the right hand side of equation (IV.4) and bond income from previous purchases of government bonds on the left.

$$y_t + rb_t = c_t + \overline{t}_t + (n+\chi)(k_t + b_t) + \dot{k}_t + \dot{b}_t \tag{III.4}$$

Competitive savings markets ensure a common return to all assets at the interest rate r. Furthermore, lump sum taxes in per-effective capita terms $\overline{t}_t$, although incomplete and thereby creating the need for debt, are nevertheless non-distortionary. The present value Hamiltonian is adjusted as follows.

$$\begin{aligned}\mathbf{H} &= U(c_t, M_t) + \lambda_t(\dot{k}_t) + \varepsilon_t(\dot{b}) \\ &= U(c_t, M_t) + \lambda_t\left[(1-\sigma)y_t - c - (n+\chi)k_t\right] + \varepsilon_t\left[\sigma y_t + rb_t - \overline{t} - (n+\chi)b_t\right]\end{aligned} \tag{III.5}$$

The solution to the dynamic problem is slightly more complicated than earlier since debt is a third endogenous state variable. Unlike in section II, the growth rates of consumption ξ and capital κ are no longer equal during the transition to the steady state. Growth of per-effective capita consumption is defined as follows. See appendix 2 for the complete derivation.

$$\begin{aligned}\xi &= \frac{1}{\Delta}\left[\frac{(1+\sigma)\frac{\varepsilon_t}{\lambda_t} + (1-\sigma)}{1+\frac{\varepsilon_t}{\lambda_t}} r + \Omega\frac{c_t}{k_t} + \Lambda\left(\frac{\alpha}{1-\beta}\kappa + \chi + n\right) - \Phi\right] \\ &\cong \frac{1}{\Delta}\left[\frac{(1+\sigma)N + (1-\sigma)}{1+N} r + \Omega\frac{c_t}{k_t} + \Lambda\left(\frac{\alpha}{1-\beta}\kappa + \chi + n\right) - \Phi\right]\end{aligned} \tag{III.6}$$

Notice the structure of the new Euler equation (III.6) has an additional two terms compared to equation (II.9). They are the growth rate of per-effective capita capital, κ and the shadow price ratio of bonds to capital, $\frac{\varepsilon_t}{\lambda_t}$. The growth of capital, bonds and consumption are jointly determined while the ratio of the shadow prices of capital and bonds,

$\frac{\varepsilon_t}{\lambda_t} \cong \frac{\varepsilon_\infty}{\lambda_\infty} = N\left(\frac{c_\infty}{k_\infty}\right)$ is determined from the steady state condition, $\xi_\infty = \kappa_\infty = 0$. The dynamic equilibrium is defined by the evolution of consumption defined by equation (III.1), the evolution of capital defined by equation (IV.7), and the relative shadow price debt consumption defined by equation (IV.8).

$$\dot{k}_t = (1-\sigma) y_t - c_t - (n+\chi) k_t \quad \textbf{(III.7)}$$

$$\frac{\varepsilon_\infty}{\lambda_\infty}\bigg|\{\xi = \kappa = 0\} = -\frac{(1-\sigma) r + \Omega \frac{c_\infty}{k_\infty} + \Lambda(\chi + n) - \Phi}{(1+\sigma) r + \Omega \frac{c_\infty}{k_\infty} + \Lambda(\chi + n) - \Phi} = N \quad \textbf{(III.8)}$$

It is important to note that the evolution of capital, equation (III.7), does not include a term for the changing level of taxation, T_t over time, which is reflective of its non-distortionary nature. Recall in section II, tax revenue was set to simply meet expenditure needs such that $T_t = \sigma Y_t$. The allowance for debt implies that $T_t \le \sigma Y_t$, which may be further defined as $T_t = \bar{\tau} Y_t \le \sigma Y_t$. Equation (III.7) is derived from differentiating the private resource constraint, equation (III.3), with respect to time while also imposing $\frac{\partial\left(\frac{T_t}{K_t}\right)}{\partial t} = 0$ such that a growing tax base does not impact the consumer's saving decision.

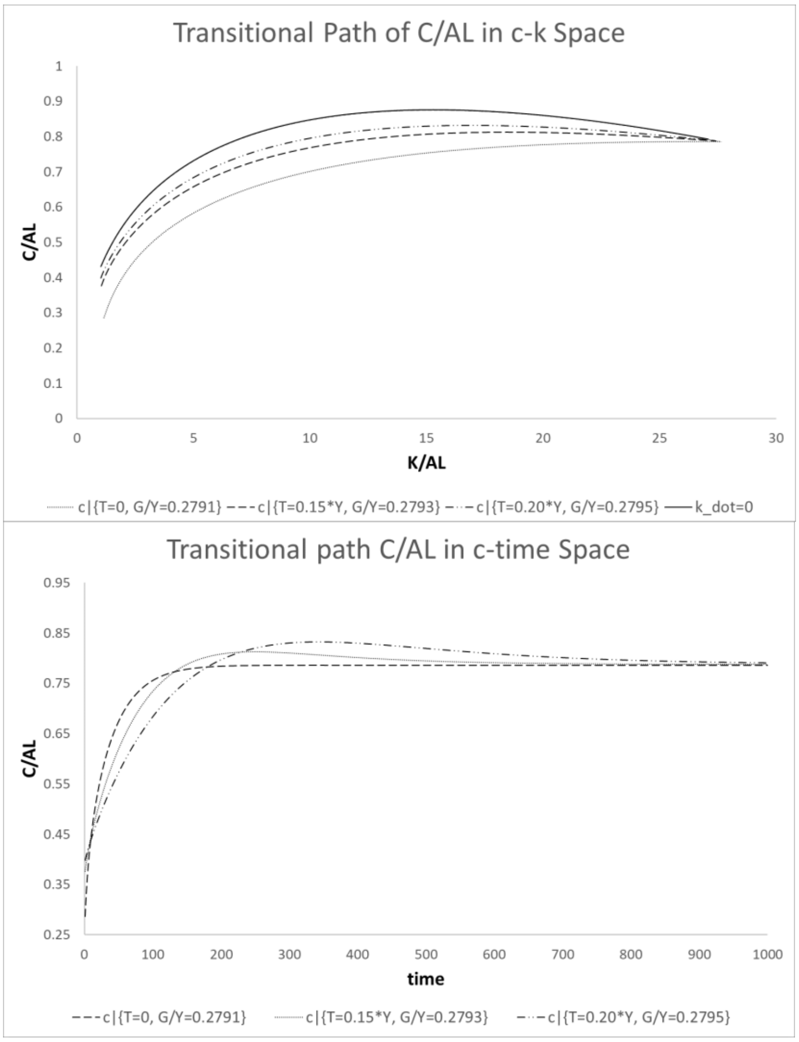

Figure 3: Saddle paths with varying degrees of public debt

A_0=1, K_0=1, L_0=1, α =0.33, β =0.25, φ =0.90, γ =0.75, ρ =0.03, θ =0.99, χ =0.02, n =0.025

We can now effectively isolate the impact of government debt across time. Figure 3 depicts the saddle paths of consumption with respect to capital as well as to time. They compare the growth paths of 100 percent debt financing of government expenditure versus partial lump sum tax financing versus a balance budget. Note that optimal government size, defined by equation (II.15), is inversely proportional, albeit marginally so, to debt levels as reflected by the falling $\frac{c_t}{y_t}$, as borrowing increases. Notice how the consumption paths converge to the same steady state. The nature of public finance, whether by debt or taxes, although irrelevant in the long run as per Ricardian Equivalence, is important to the transition to the steady state. In other words, during the transitional dynamics- in the short to medium term- debt matters.

For any fiscal policy, there must exist a predictable transition path to the long run equilibrium shared by them all. As a practical convention, we define the short run as immediately following t=0 and the medium run as the period around $t \approx t\left|\{c = c_{\max}\}\right.$. In figure 3, the medium run is between 100 to 300 periods, inversely proportional to relative debt. Debt stimulates growth as can be seen in figure 3b by the increasing slope of the consumption path as taxes fall and debt rises. But improved early growth rates are paid for by a lower medium-term consumption. Note that the growth paths with respect to capital in figure 3a are uniformly above those with greater debt finance. In figure 3b, as debt falls, while growth declines as the consumption path has a lower slope as relative debt rises, the path with the greatest debt eventually achieves highest level of per capita consumption. The figure demonstrates the dynamic impact of debt on intertemporal consumption. Greater public debt implies greater private saving and higher growth in the short run. Eventually, some 300 periods from the model's initiation, the cost of the debt is felt in terms of consumption levels.

THE MODEL - PUBLIC GOODS, TAXATION AND DEBT

Suppose again that relative government spending, σ, is policy driven and independent of revenue. Imposition of a balanced budget on such a government necessarily implies that expenditure ultimately determines revenue and the only choice remaining to the government is one of tax mix. Although an interesting exercise which has been explored in both static[10] and dynamic settings, it is conceptually little more than a benchmark in the face of the stylized facts pertaining to the short run versus long run management of public budgets.

In the long run, government revenue and expenditure must be jointly determined to some extent. The accepted logic in public economics; how taxes are set depends in general on how revenue is spent and the impact of said expenditure on private agents. Although the importance of analyzing the tax mix empirically has been extensively demonstrated (See Kneller et al., 1999; Bleaney et al., 2001; Nikos, 2009; Debotoli and Gomes, 2012), none establish clear causality in either direction.

Anecdotally, government expenditure decisions are made independently of the current revenue considerations and with only dubious link to future revenue choices. Although today's spending decisions can certainly impact future revenue choices, the reliance on deficits and debt makes the causal link between expenditure and revenue difficult to establish analytically in the short run.

There are several endogenous growth models that explicitly consider the tax mix between indirect and direct taxation. Contributions by Barreto and Alm (2003), Gomez (2007), Agenor and Neanidis (2014) and Zhang, Ru and Li (2016) explicitly consider the welfare and growth implications of tax structure but do so under the assumption of balanced budgets. These authors by construction create bidirectional causality between government revenue and expenditure.

We presume that government is simply incapable of an integrated optimal policy. The government chooses a relative level of public production, optimally or not, and an arbitrary tax mix. We assume tax revenues do not meet expenditure such the government incurs a budget deficit. Although the government could run budget surpluses to ultimately be a net lender, we do not explicitly analyze this circumstance. The government budget constraint is met by public borrowing from consumers who are indifferent between saving for capital accumulation and lending money to the government.

The model is structurally similar to Turnovsky (1996) save for the following important distinctions. First, we explicitly include labour in our analysis. The added dimensionality allows for non-exclusivity of public goods in utility, a richer characterization of the equilibrium saddle path as well as allows for more interesting extensions. Labour, within the Cobb-Douglas production function, is subject to diminishing returns which ultimately implies long run steady states in per-effective capita consumption, capital and debt characterized by a modified golden rule. Second, Turnovsky's social planner recognizes the optimal size of government as well as recognizes it can be achieved with or without a balanced budget. Since the transversality condition guarantees the transitional dynamics of debt to be irrelevant in the long run, debt has no impact of the steady state equilibrium, and is therefore dropped from his model. Our methodology, while confirming the long run irrelevance of debt, demonstrates the non-monotonic nature of debt on consumption and saving throughout the transition to the long run steady state.

[10] A review of issues pertaining to consumption tax's role in the tax mix can be found in Atkinson and Stiglitz (1976).

The inclusion of debt does not fundamentally change the consumer's problem. She still maximizes utility given a return on saving. Before savings trivially becomes new capital however, it is divided into capital accumulation and government bonds, which pay the same market return, *r*. The consumer's private resource constraint, where, τ is the income tax rate and ω is the consumption tax rate, is defined as follows.

$$\left(Y + rB_t\right)\left(1-\tau\right) = C_t\left(1+\omega\right) + \dot{K}_t + \dot{B}_t \ : \text{PRC} \qquad \textbf{(III.9)}$$

Consumers are taxed on both production income and bond income. The government's budget constraint is therefore defined as follows.

$$\sigma Y_t = \tau\left(Y_t + rB_t\right) + \omega C_t + \dot{B}_t - rB_t \ : \text{GBC} \qquad \textbf{(III.10)}$$

The evolution of bonds, equation (V.4), is determined by the relationship among the private resource constraint, equation (III.9), the government budget constraint, equation (III.10), and the aggregate resource constraint, equation (V.3).

$$\dot{K}_t = \left(1-\sigma\right)Y_t - C_t \ : \text{ARC} \qquad \textbf{(III.11)}$$

$$\dot{B}_t = rB_t\left(1-\tau\right) + \left(\sigma - \tau\right)Y_t - \omega C_t \qquad \textbf{(III.12)}$$

Equation (III.12) may be rewritten in terms of an effective tax rate, $\hat{\tau} = \hat{\tau}\left(\tau, \omega, s\right)$ which is a function of the saving rate, *s*, and defined by equation (V.5), to yield the evolution of bonds as function of the effective tax rate relative to the government size, equation (V.6).

$$\hat{\tau} = \left[\frac{\omega}{1+\omega}\left(1-\tau\right)\left(1-s\right) + \tau\right] \qquad \textbf{(III.13)}$$

$$\dot{B}_t = \left(\sigma - \tilde{\tau}\right)Y_t + \left(1-\tilde{\tau}\right)rB_t \qquad \textbf{(III.14)}$$

The effective tax rate is particularly useful since it defines the iso-tax relationship of possible combinations of τ and ω such that, for any given effective tax rate there exists a continuum of possible tax mixes to achieve it. As expected, the relationship between τ and ω is defined by $\frac{\partial \tau}{\partial \omega} < 0$ and $\frac{\partial^2 \tau}{\partial \omega^2} > 0$ for any given $\hat{\tau}$ [11] where $\hat{\tau}$ defines the tax substitution between income taxes and consumption taxes. Furthermore, for any given rate of public expenditure, σ, it can be shown that combinations of τ and ω that satisfy $\hat{\tau}\left(\tau, \omega\right) \cong \sigma$ optimize welfare, while any tax mix that achieves an effective rate equal to the productivity of capital, $\hat{\tau}\left(\tau, \omega\right) \cong \beta$, maximizes steady state consumption. Recall that welfare maximization necessarily includes public goods such that optimal consumption may not be the highest consumption achievable at the steady state. For any given effective tax rate, $\frac{\partial \tau}{\partial \omega}$ is declining in the saving rate, *s*. Therefore, the lower the saving rate, the more effective is the tax substitution of income taxes by consumption taxes. The actual tax mix in isolation, defined by $\partial \hat{\tau} = 0$ for any σ, primarily impacts the transition. Higher incomes taxes deter growth in favor of higher per capita consumption while higher consumption tax rates lower medium-term per capita consumption and indirectly favor faster growth.

This gives us a clear analytical method to consider the concept the optimal tax mix as locus of possibilities. Recall the utility maximizing rate of public expenditure defined by equation (II.15). Setting optimal expenditure equal to the effective tax rate yields the locus of welfare maximizing tax mixes that jointly optimizes welfare given optimal government expenditure. An important caveat to the welfare optimality condition, $\hat{\tau} = \hat{\tau}\left(\tau, \omega\right) \cong \sigma = \sigma *$, is the possibility, albeit not necessarily analytically defined, to raise growth and consumption without loss in welfare by skewing the tax mix toward income taxes. Although the difference on steady state consumption across tax mixes is quite marginal, the relationship is nevertheless systematic.

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[11] Given that $\hat{\tau} = \hat{\tau}\left(\tau, \omega, s\right)$, it is important to also note that $\frac{\partial s}{\partial \omega} < 0$ and $\frac{\partial s}{\partial \tau} < 0$ at any time *t*.

The representative agent maximizes lifetime welfare with respect to consumption, capital saving, and bond saving. Capital evolves simultaneously to bonds from the competition for savings and intertemporal substitution as defined in the aggregate resource constraint. The evolution of consumption is defined as follows, where $N = \frac{\varepsilon_\infty}{\lambda_\infty}$.

$$\xi \cong \frac{1}{\Delta}\left[\frac{\left[\sigma+\omega-\tau(1+\omega)\right]N+(1-\sigma)}{1+N\omega}r+\Omega\frac{c_t}{k_t}+\Lambda\left(\frac{\alpha}{1-\beta}+\chi+n\right)\kappa-\Phi\right] \quad \textbf{(III.15)}$$

Notice the parallels of the growth in consumption with distortionary taxes and debt, equation (III.15), versus the growth in consumption in the absences of proportional taxes, equation (III.6). Only the first term, the marginal product of capital that includes income and consumption taxes, has changed.

The evolution of capital per capita is simply $\kappa = \frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - n - \chi$, given the aggregate resource constraint, equation (III.11). Rearranging the expression for growth to define the relative shadow prices of bonds and capital given the steady state condition, $\xi_\infty = \kappa_\infty = 0$, allows us to solve the circularity problem in equation (III.15).

$$\left.\frac{\varepsilon_\infty}{\lambda_\infty}\right|\{\xi=\kappa=0\} = \frac{(1-\sigma)r+\Omega\frac{c_\infty}{k_\infty}+\Lambda(\chi+n)-\Phi}{-\omega\left\{(1-\tau)r+\Omega\frac{c_\infty}{k_\infty}+\Lambda(\chi+n)+\Phi\right\}-(\sigma-\tau)r} = N \quad \textbf{(III.16)}$$

Equations (III.11), (III.12) and (III.15) represent the evolution of the three endogenous state variables. Equation (III.16) is the binding constraint that defines the shadow price of debt versus consumption in the general equilibrium. The system of three equations can be theoretically solved iteratively at any point along the saddle path given the $\lim_{t\to\infty}\xi_t = \lim_{t\to\infty}\kappa_t = 0$ end point condition defined by equation (III.16).

Some simple stylized facts may be considered and necessarily compared. First, the effective tax rate of developed countries is generally higher than that of less developed ones. It follows that for any given rate of government expenditure, whether optimal or not, less developed economies will necessarily have greater debt. Furthermore, assuming the tax mix in developed economies is skewed toward income taxes, comparing developed to less developed fiscal policies is to compare a higher tax base generated from income taxes and relatively less debt to a lower tax base generated from consumption taxes and relatively more debt. Adding to that is the possibility of greater congestion in developing countries.

SIMULATIONS

The analytical design and modeling technology used to create the simulations is explicitly suited to investigate the equilibrium transitional dynamics of the short run and medium run, in addition to the more traditional consideration of the long run steady state. We compare the theoretically estimated consumption path of a country to its true path. Given a reasonable fit, we can use the simulation to consider the impact of shocks and consequent potential policy implications in a dynamic setting.

Imagine two hypothetical economies that generate two unique saddle paths of consumption. Although the initial conditions, defined by the starting values of A_{i0}, K_{i0}, L_{i0} and B_{i0}, are irrelevant to the eventual steady state equilibrium, the starting values necessarily make the consumption path unique. The idiosyncratic nature of the saddle path as a function of its starting values extends to include the hypothetical comparison of two countries that are identical in every way except the initial conditions underlying their respective consumption paths.

If two countries share the same coefficient values, irrespective of their starting capital, labour, technology or debt, they will converge to the same steady state per-effective capita consumption, capital, and debt. Even given the extra dimensionality of debt, the variation in steady state equilibria generated herein, like other endogenous growth models, is relatively narrow as compared to the variation in the real data. This is one of the main criticisms of endogenous growth theory as an analytical tool (Jones, 1995).

Consider instead the practical hypothesis that no country, including the United States, is anywhere near its respective long run equilibrium. Suppose each country *i* is exactly 70 years into its idiosyncratic transition to a unique steady state in the distant future. Imagine starting the growth clock of the simulation at t=0 with each country reports comparable 1950 values for capital, labour, technology, and debt to serve as the initial conditions. Suppose the population growth rate, n_i , the rate of Harrod neutral technological change, χ_i , the income tax rate, τ_i , the consumption tax rate, ω_i, the government spending rate, σ_i , and the congestion rate, γ_i , are country specific but time invariant. All other coefficient values are identical across countries. Table 1 summarizes the data.

We include total factor productivity (TFP), represented by a_i, such that the production function that we actually simulate, equation (VI.1) is only a slight variation of equation (II.7).

$$Y_{it} = a_i K_{it}{}^{\alpha} G_{it}{}^{\beta} \left(A_{it} L_{it} \right)^{1-\alpha-\beta} \tag{III.17}$$

It can be readily shown that if total factor productivity is constant in the long run, such that $\lim_{t\to\infty} \frac{\partial a_t}{\partial t} = 0$, the saddle path of consumption will converge to a steady state equilibrium. More specifically, if TFP is constant, then it has only a level effect within the equilibrium and is therefore excluded from within the literature that is generally concerned with the steady state. But if transitions are the relevant comparable outcome, even if constant, TFP plays a significant role.

Table 1A

Country specific tax rates, public expenditure rates and state variable growth rates

	*Public Finances(1990-2015 avg.)**			*cwtfp(avg.)***		*χ : Tech.Growth(yrly.avg.)****		*n: Pop.Growth(yrly.avg.)*	
country	*τ*	*ϖ*	*σ*	*1950-2015^*	*1970-2015*	*1950-2015^*	*1970-2015*	*1950-2015^*	*1970-2015*
Australia	14.56%	6.72%	25.06%	0.8472	0.8453	0.42%	0.37%	1.62%	1.40%
Burkina Faso	1.22%	3.00%	12.72%	0.3938 (60)	0.3948	0.37%	0.44%	2.31%	2.60%
Switzerland	1.75%	4.31%	17.47%	0.8425	0.8361	0.35%	0.33%	0.91%	0.67%
Chile	4.15%	10.19%	18.39%	0.6930 (51)	0.6578	0.76%	0.82%	1.61%	1.40%
Côte d'Ivoire	1.78%	2.33%	14.47%	0.7311 (60)	0.7132	0.81%	0.97%	3.57%	3.40%
Dominican Republic	2.45%	6.28%	11.74%	0.7740 (51)	0.7787	1.03%	1.27%	2.30%	1.93%
Spain	4.87%	6.48%	20.24%	0.9315	0.9841	0.68%	0.77%	0.77%	0.70%
Guatemala	2.52%	5.71%	11.11%	0.8621	0.8311	0.71%	0.97%	2.55%	2.38%
India	3.22%	3.88%	15.35%	0.3366	0.3357	0.94%	1.24%	1.96%	1.94%
Ireland	9.42%	13.56%	35.25%	0.8499	0.8832	0.57%	0.58%	0.72%	1.04%
Jamaica	8.78%	6.00%	26.11%	0.5039 (53)	0.4885	0.62%	0.58%	1.17%	0.96%
Japan	5.31%	4.46%	16.52%	0.6358	0.6834	0.68%	0.54%	0.65%	0.46%
Republic of Korea	3.92%	6.35%	17.56%	0.4877 (54)	0.5440	1.13%	1.37%	1.44%	1.04%
Sri Lanka	2.03%	8.60%	21.02%	0.6998	0.6537	1.02%	0.91%	1.59%	1.15%
Morocco	5.27%	10.40%	23.94%	0.9216	0.8783	0.90%	1.34%	2.04%	1.46%
Mexico	2.25%	3.89%	16.37%	0.8677	0.8131	0.88%	0.85%	2.34%	1.57%
Malaysia	5.60%	4.07%	18.65%	0.4675 (55)	0.4882	1.38%	1.54%	2.44%	2.35%
Peru	3.34%	7.44%	17.71%	0.5701	0.5372	1.07%	1.18%	2.18%	1.94%
Philippines	5.08%	4.24%	15.40%	0.4828	0.4909	1.12%	0.99%	2.57%	2.36%
Paraguay	0.41%	1.92%	13.24%	0.5971 (54)	0.5971	0.93%	0.97%	2.32%	2.23%
Singapore	4.09%	5.67%	14.43%	0.7300 (60)	0.7393	1.70%	1.77%	2.23%	2.19%
Thailand	4.87%	9.48%	15.51%	0.3961	0.4207	1.23%	1.33%	1.93%	0.93%
United States	5.62%	0.43%	21.48%	1.0000	1.0000	0.57%	0.46%	1.12%	0.95%
South Africa	12.34%	10.39%	29.91%	0.7161	0.7225	0.77%	0.92%	2.18%	2.00%

* = World Development Indicators (2017), ** = Welfare relevant total factor productivity (USA=1) from Penn World Tables (See Feenstra, et.al., 2015),

*** Human capital index based on PWT 9.0, ^ 1950 base year unless otherwise noted in brackets

Table 1B

Country starting values of capital, labour and harrod neutral techology

country	*K(1950)^*	*L(1950)^*	*A(1950)^*	*K(1970)*	*L(1970)*	*A(1970)*
Australia	3.51	8.39	2.67	7.99	12.84	2.97
Burkina Faso	0.03 (60)	5.16 (60)	1.01 (60)	0.03	5.62	1.01
Switzerland	4.50	4.62	2.94	8.80	6.17	3.18
Chile	0.79 (51)	6.38 (51)	1.89 (51)	1.62	9.56	2.13
Côte d'Ivoire	0.14 (60)	3.36 (60)	1.04 (60)	0.39	5.24	1.04
Dominican Republic	0.16 (51)	2.46 (51)	1.38 (51)	0.32	4.50	1.50
Spain	4.53	28.15	1.87	16.97	33.98	2.05
Guatemala	0.20	3.17	1.18	0.72	5.62	1.20
India	6.34	369.67	1.13	21.88	553.58	1.18
Ireland	0.62	2.96	2.16	1.41	2.95	2.40
Jamaica	0.42 (53)	1.40 (53)	1.77 (53)	0.90	1.88	1.99
Japan	7.63	84.27	2.29	41.65	104.93	2.80
Republic of Korea	1.34 (54)	21.24 (54)	1.84 (54)	2.06	32.21	1.98
Sri Lanka	0.47	7.45	1.48	0.87	12.49	1.95
Morocco	1.09	9.40	1.03	1.64	16.00	1.08
Mexico	3.12	28.08	1.53	13.30	52.03	1.73
Malaysia	0.51 (55)	7.24 (55)	1.31 (55)	0.94	10.80	1.50
Peru	0.52	7.72	1.39	1.76	13.34	1.64
Philippines	1.18	19.60	1.29	3.44	35.80	1.71
Paraguay	0.09 (54)	1.64 (54)	1.45 (54)	0.15	2.47	1.65
Singapore	0.35 (60)	1.64 (60)	1.46 (60)	0.77	2.07	1.66
Thailand	0.90	19.93	1.21	3.67	36.88	1.43
United States	88.92	155.64	2.58	179.95	209.59	3.06
South Africa	2.11	13.66	1.65	6.83	22.84	1.79

So. Penn World Tables (2017), ^ 1950 base year unless otherwise noted in brackets

We compare the theoretical growth paths generated by the simulations given comparable initial values for capital, labour and technology for twenty-four developed and developing countries. We derive consistently approximated average income tax rates, consumption tax rates and public expenditure rates for 1990 to 2015 from the World Development Indicators (2019)[12] and use these estimated rates throughout the simulations. Growth rates of human capital and labour are their annual averages. The data is presented in Tables 1A and 1B. Note that the data for total factor productivity are relative to the United States.

The model requires initial state conditions for technology, capital, labour and debt. Given our assumption of Harrod neutral technology in the production function, we assume the labour enhancing technological factor represents the quality of human capital. We use the human capital index from the Penn World Tables 9.0 provided by Feenstra, et.al. (2015), taking its annual average growth rate from 1950 to 2015, as a proxy for Harrod neutral technology factor A_t in equation (III.17). We assume that total factor productivity, also from the above same source, is fixed at the average level from 1950 to 2015.

Our initial dataset of consistent tax and government spending data is comprised of 104 countries. Of these, given their respective initial state conditions, 80 countries exhibit too much spending relative to tax receipts for the model to converge along a strictly positive consumption path. Most EU countries fell into the category as did countries with significant public income outside taxation.

Table 2

Predicted versus actual consumption per effective capita at year 2015

	Actual^		Predicted from base year = 1950~			Predicted from base year = 1970		
country	***P_cC/AL(2015)***	**%USA(2015)**	**Predicted C/AL**	**% of USA**	**Difference**	**Predicted C/AL**	**% of USA**	**Difference**
Australia	10798.35	90.30%	0.67	89.64%	0.66% ***	0.55	88.53%	1.78% ***
Burkina Faso	418.85	3.50%	0.14 (60)	6.50%	-2.99% ***	0.12	19.36%	-15.85% **
Switzerland	12771.45	106.80%	0.61	82.09%	24.71% *	0.49	78.30%	28.51% *
Chile	3209.34	26.84%	0.42 (51)	56.58%	-29.74% *	0.33	52.84%	-26.00% *
Côte d'Ivoire	653.75	5.47%	0.33 (60)	15.60%	-10.13% ***	0.32	51.06%	-45.59%
Dominican Republic	2048.37	17.13%	0.45 (51)	61.24%	-44.11%	0.40	63.67%	-46.54%
Spain	6067.59	50.74%	0.74	99.23%	-48.49%	0.65	103.80%	-53.06%
Guatemala	2020.40	16.90%	0.55	73.19%	-56.29%	0.43	69.17%	-52.27%
India	545.23	4.56%	0.10	14.01%	-9.45% ***	0.08	13.55%	-8.99% ***
Ireland	8523.72	71.28%	0.10	13.09%	58.19%	0.07	11.34%	59.94%
Jamaica	1957.77	16.37%	0.21 (53)	29.61%	-13.24% **	0.17	27.47%	-11.10% **
Japan	6840.22	57.20%	0.41	55.40%	1.80% ***	0.38	60.85%	-3.65% ***
Republic of Korea	5046.16	42.20%	0.21 (54)	29.77%	12.43% **	0.22	35.08%	7.12% ***
Sri Lanka	1091.18	9.13%	0.34	45.18%	-36.06%	0.26	41.95%	-32.83%
Morocco	1248.30	10.44%	0.53	71.44%	-61.00%	0.42	66.88%	-56.44%
Mexico	2436.97	20.38%	0.35	47.35%	-26.97% *	0.42	67.79%	-47.41%
Malaysia	2275.16	19.03%	0.17 (55)	25.19%	-6.17% ***	0.16	25.98%	-6.95% ***
Peru	1733.29	14.49%	0.26	34.82%	-20.32% **	0.20	32.24%	-17.75% **
Philippines	907.51	7.59%	0.19	25.84%	-18.25% **	0.17	27.85%	-20.26% **
Paraguay	1315.21	11.00%	0.27 (54)	39.45%	-28.45% *	0.24	37.91%	-26.91% *
Singapore	7105.59	59.42%	0.38 (60)	48.17%	11.25% **	0.35	56.81%	2.61% ***
Thailand	1469.61	12.29%	0.15	19.80%	-7.51% ***	0.14	23.01%	-10.72% **
United States	11957.87	100.00%	0.74	100.00%	0.00%	0.62	100.00%	0.00%
South Africa	1659.65	13.88%	0.33	43.66%	-29.78% *	0.28	44.60%	-30.72%

^ So. Penn World Tables (2017), ~ 1950 base year unless otherwise noted in brackets, *** ABS(Difference)<0.1, ** ABS(Difference)<0.2, * ABS(Difference3)<0.3

We record the simulations results for each country for up to 65 periods using 1950 to 2015[13] data. We then compare the predicted point along the consumption path 65 periods from the model's initiation to the actual 2015 data. The results are presented in table 2. For example, the value of per effective capita consumption (P_cC/AL) in Australia in 2015 was 10798.35, representing 90.30 percent of US per effective capita consumption. Similarly, the value of per effective capita consumption in Burkina Faso in 2015 was 3.50 percent of USA. The model predicts per effective capita consumption in Australia in 2015 would have been 89.64 percent of USA given its 1950 initial conditions. Similarly, the model predicts per effective capita consumption in Burkina Faso would have been 6.50 percent of USA. Of the 23 countries besides the USA, predicted relative consumption in 7 countries are within 10 percent of their actual values, 12 of the 23 predictions are within 20 percent of actual and 17 of the 23 predictions are within 30 percent of their actual values.

We may consider theoretical shocks, whether permanent or transitory, that occur during a country's transition to their respective steady state, which can serve to demonstrate the economy-wide dynamic impact of said shocks. The model

[12] Exogenous rates represent the averages from 1990 to 2015 derived from the WDI indicator codes as follows, τ = GC.TAX.YPKG.RV.ZS * GC.TAX.TOTL.GD.ZS / 10000, ω = GC.TAX.GSRV.RV.ZS * GC.TAX.TOTL.GD.ZS / (NE.CON.TOTL.ZS * 100), σ = GC.XPN.TOTL.GD.ZS / 100

[13] In the unbalanced dataset, countries commence their respective simulations at the earliest year pendent upon data availability. See table 2.

framework allows for any number shocks to be represented such that we may also consider policy responses and assess their dynamic impact moving forward. The reporting technique allows us to assess the dynamic impact of the shock and subsequent possible policy responses on any aspect of the economy. We can perfectly track shocks and policy responses from occurrence through the short run transition and onto the long run steady state.

GOVERNMENT DEFICIT SHOCK

We consider two fiscal policy responses to an economy wide productivity shock to effective labour. We time the productivity shock with the Covid-19 pandemic of 2020 or exactly seventy periods into the simulation for each country. In response to the shock, governments can provide temporary income support (IC_{it}) and or a costly vaccine (IM_{it}), delivered as a public service. Either policy is freely provided to consumers by the government and paid for entirely with debt. Our modeling of the pandemic shock follows Gori, et al (2022) and is identical to Alm and Barreto (2024).

We assume the shock, represented by μ_t, affects production at some time $t = \bar{t}$ as follows.

$$Y_{it} = a_i K_{it}{}^{\alpha} G_{it}{}^{\beta} \left(\mu_t A_{it} L_{it}\right)^{1-\alpha-\varphi} \left| \begin{array}{l} \mu_t \le 1 \ \forall \ t \ge \bar{t} \\ \mu_t = 1 \ \forall \ t < \bar{t} \end{array} \right. \quad \textbf{(IV.1)}$$

The scale of the shock evolves simply from its initial appearance such that $\frac{\partial \mu_t}{\partial t} > 0$ and $\frac{\partial \dot{\mu}_t}{\partial t} < 0$ to its limit of $\mu_\infty = 1$. For the sake of brevity, we assume an exogenous recovery rate, δ, that determines the percentage of the previous period's loss that is made up naturally in the subsequent period. The depth of the shock dissipates naturally by the following rule.

$$\mu_{t+1} = \left(1-\delta\right)\mu_t + \delta \ \forall \ t > \bar{t} \quad \textbf{(IV.2)}$$

We assume countries respond to the crisis immediately with income support, IC_{it}, whose scale is defined exogenously as a percentage of GDP and determined annually. The income support enters the private sector via the private resource constraint, rearranged to define consumption.

$$C_{it} = \frac{\left(Y_{it} + rB_{it}\right)\left(1-\tau_i\right) - S_{it} + IC_{it}}{1+\omega_i} \quad \textbf{(IV.3)}$$

In addition, governments provide free vaccinations which is represented as a temporary increase provision of nonproductive public goods, subject to corruption, in the utility function.

$$M_{it} = \sigma_i Y_{it}^{\gamma_i} + IM_{it} \quad \textbf{(IV.4)}$$

The extra government expenditure is accounted for through the government budget constraint and appears analytically in the evolution of bonds, a version of equation (III.12).

$$\dot{B}_{it} = \left[\sigma_i - \omega_i\left(1-\sigma_i\right) - \tau_i\right] Y_{it} + \left(1-\tau_i\right) rB_{it} + \omega_i \dot{K}_{it} + IC_{it} + IM_{it} \quad \textbf{(IV.5)}$$

The impact of the shock is significant and persistent. Figure 4 shows the simulation for the United States from 1950. We assume the initial depth of the shock to be $\mu_{70} = 0.90$ and the natural recovery rate of $\delta = 0.15$. The six simulations in Figure 4 depict no shock, the shock with no response, the shock followed by 1 percent of GDP as income support for one year, the shock followed by 2 percent of GDP as income support for one year, the shock followed by 5 percent of GDP as income support for one year, and the shock followed by 10 percent of GDP as income support for one year. Notice that the productivity shock can be offset completely in the short to medium run by income transfers but at a cost in the longer term.

Income support, although possibly leading to higher initial welfare necessarily results in declining relative consumption after the initial shock. The scale of that decline is determined by the depth of the income support. Notice that 1 percent of US GDP in this simulation completely offsets the shock with only a marginal impact on the longer-term consumption path while greater stimulus leads to a peak in consumption followed by a decline. The model suggests that there exists some level of temporary income support that will offset the impact of a labour productivity shock on the level of consumption without the future cost in declining consumption growth. The model demonstrates the dynamic consumption cost of too much deficit financed income support. It is simply Ricardian equivalence in action. The debt induced consumption spike immediately following the pandemic is ultimately paid for down the road with lower relative consumption from around 2060.

The productivity impact of deficit financed income support is further made clear by looking at the expansion path of output per-effective capita. Although income support can offset consumption, it ultimately exacerbates the productivity shock by lowering the incentive to work as the return to labour falls relative to the return on capital. This is depicted in the expansion path of the wage-rental ratio. Note that growth is negative in the years following the income support.

Figure 4: Deficit Financed Income Support = 1, 2, 5 or 10 percent of GDP to private income

A_0=2.58, K_0= 88.92, L_0= 155.64, α =0.33, β =0.25, φ =0.90, γ =0.75, ρ =0.03, θ =0.99, χ =0.02, n =0.025 τ=0.0562, ω = 0.0043, σ =0.2148

The driving analytical differences amongst countries within the panel are public finance rates, total factor productivity, human capital growth and population growth. Yet the model is rich enough to conjecture about the differences between more and less developed countries in the face of the productivity shock. For example, while we assume a common congestion rate across all countries, anecdotal evidence suggests this is certainly wrong. We also assume the elasticity of substitution between private consumption and public goods is common across countries. Yet if we interpret this elasticity to reflect one's reliance on public goods, it is likely that this also varies across countries.

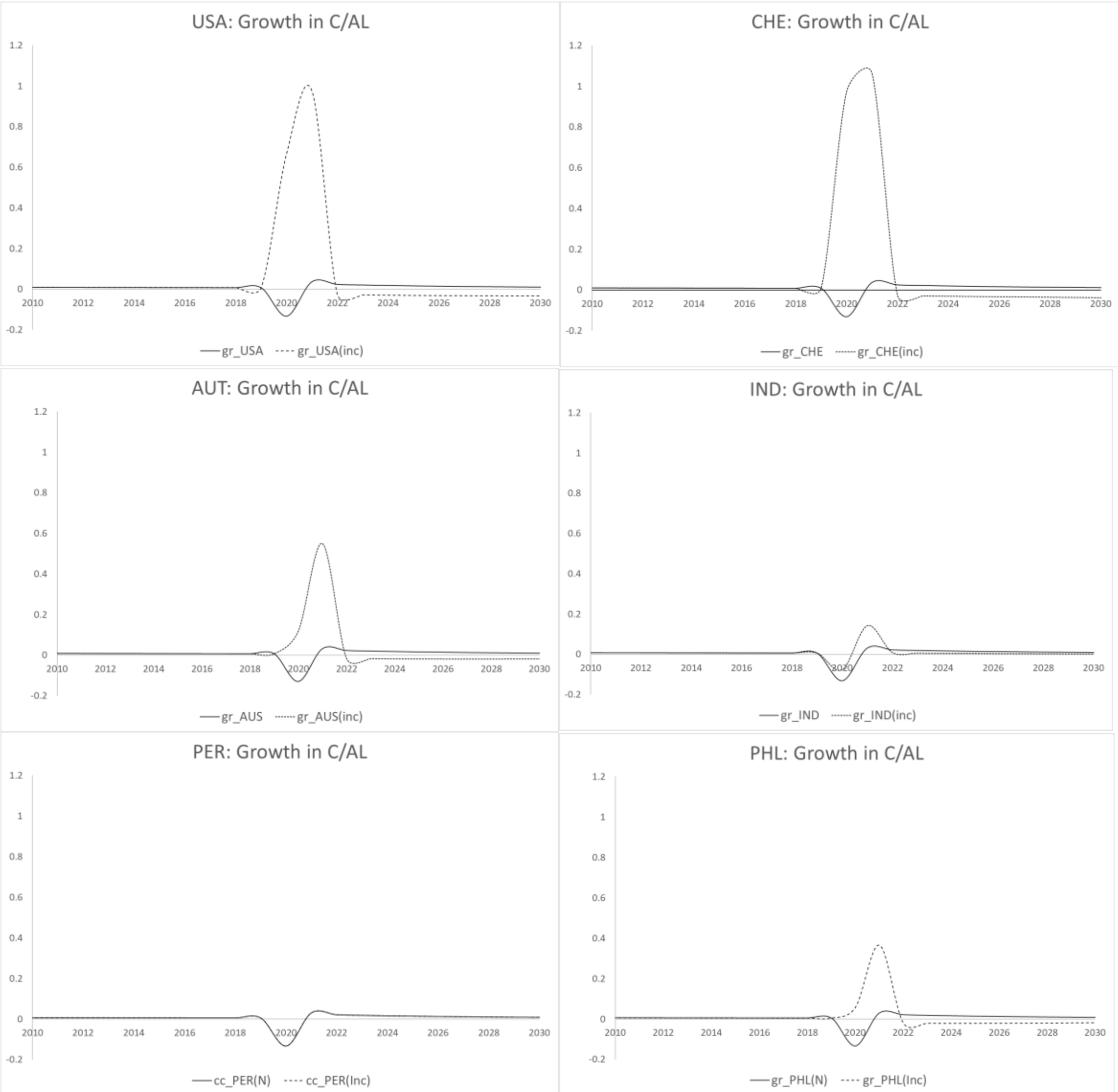

Figure 5: Deficit Financed Country Specific Income Support

Figure 5 depicts the growth impact of the shock to six of the countries sampled. Although the relative depth of the shock is similar across countries, their individual responses are not. USA is assumed to provide 10 percent of GDP in 2020 as income support compared to Switzerland's 15 percent, Australia's 5.4 percent, Philippines' 3.9 percent, India's 1 percent, and Peru's 0 percent (IMF, 2020).

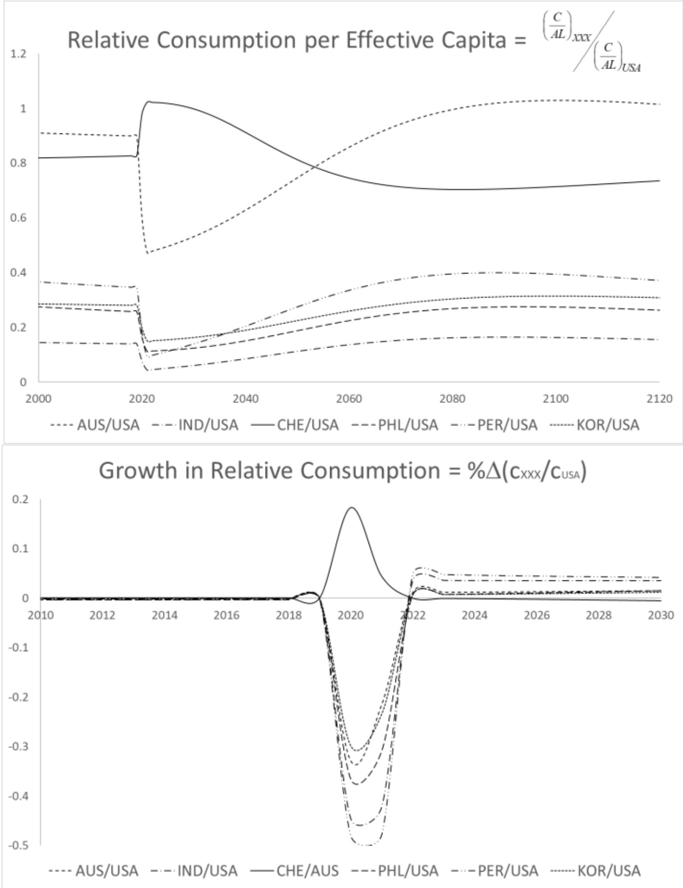

Figure 6: Consumption per effective capita relative to the USA given individual country responses

The Swiss are the only ones whom we would expect their per-effective capita consumption to decline relative to their American counterparts in the years following 2020. Figure 6 directly compares the relative consumption paths of each country. Every country suffers an initial negative shock where the impact appears greater in the countries with less fiscal stimulus. After the initial shock, most countries can expect their per-effective capita consumption to effectively increase faster than that of the United States.

The model suggests many countries which are unwilling or unable to provide as much relative income support as the United States will suffer a noticeable drop in the relative level of consumption versus 2019 but immediately see greater growth in consumption than before the pandemic. If we assume countries such as USA and Switzerland, whose fiscal responses are highest, are both overshooting their optimal level of income support, the model predicts these two countries will not suffer a level shock to consumption in 2020, but instead will likely see declining consumption relative to the rest of the world in the years following.

CONCLUSION

We make three contributions to literature. The first is the analytical framework, which provides a closed form solution to a three-dimensional RCK model with unproductive public goods in the utility function and a productive public sector in the production function that are both simultaneously finance by taxes and or debt. The second contribution is our methodology to represent the transitional dynamics of the state space by exploiting advanced software applications, explicitly suited to modeling continuous time mathematics. The third is the model's application as a predictive tool to investigate the future implications of today's fiscal policy decisions.

The idiosyncratic government responses to the Covid-19 pandemic will likely lead to an equal variation in observed outcomes of said policies. Within a closed form two-sector endogenous growth model, using simulations based on real data, we predict the welfare effects of the pandemic at hypothetical year 2020 given income support to the private sector financed entirely by government debt. We trace the impact of the public debt, amassed by the government's response to the crisis, on the economy through the short and medium term. We demonstrate the transitional dynamic trade-off of public deficits- more consumption today at the expense of declining consumption tomorrow- while maintaining Ricardian equivalence in the long run.

Our model indicates there can be too much debt financed income support. If a country provides 'too much' deficit financed income support in the first year of the pandemic, consumption will rise discernably relative to pre-pandemic levels at the expense of relative falling consumption in the years following. Alternatively, the model also suggests there exists some degree of direct income support financed entirely by debt that offsets the pandemic's immediate impact on consumption without the subsequent fall in growth.

The predictive aspect of the model is by construction relative to the United States. Under the presumption the US rate of support is 'too high', most other countries' consumption growth, post 2020, will appear to rise relative to the US. However, the model does not necessarily suggest the United States has in fact overshot its 'best' rate. That assessment is left to time.

We have only considered one extension to a model that is flexible enough to consider any number of treatments. For example, one might reassess the many conclusions drawn from the government and growth literature that assume balanced budgets. One could consider the impact of debt on fiscal equalization, as in Cyrenne and Pandey (2015), or on the quality of the public infrastructure as in Agenor (2007). We consider a single production function. Instead, suppose there are multiple sectors that use public services differently as in Felice (2016). Considering our representative agent saves both capital and bonds, financial intermediation as in Eggoh and Villieu (2014) could be investigated. While we present a deterministic model, stochastic elements could also be introduced.

Adding dimensionality to endogenous growth models adds great richness at the expense of complicated formulae to make analysis of the transitional and equilibrium properties difficult. Nevertheless, the models can be constructed by piecing together component parts, as we do here. With any of the extensions suggested above, the part of the general equilibrium investigated can be thought of as a sort of component part that can be substituted in or out of the basic analytical framework. The interaction of the parts could be investigated without necessarily having to analytically define likely indeterminacies one may encounter while simulating the equilibria. We propose to allow advanced software applications, explicitly suited to continuous time mathematics and necessarily capable of solving far more complicated general equilibrium systems than what are typically defined in economic growth models, to serve as a medium for analysis.

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