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NATURAL LOG OF ZERO IS A COMPLEX NUMBER

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ABSTRACT
We take the definition for $n!$, which takes on integer values $\forall n \geq 1$, and, through a conjecture, generalize it to $\forall n > -\infty$, from which we demonstrate that $ln(0)$ is a complex number.

KEYWORDS:
Mathematics, Complex numbers, Logarithm of Zero

INTRODUCTION

The characterization of the natural log of zero (i.e. $ln(0)$), as given in all mathematics texts and references, is that it is "undefined", with no additional details of what this really means. This characterization is trivially obvious because $\lim_{x\to 0} ln\, x \to -\infty$. The question we try to address here is how does the trajectory of the approach to $-\infty$ occur?

There is currently no mention in the literature of how the approach to $-\infty$ happens, i.e. whether the path follows the real axis or it goes through the complex plane. All we are told is that

$$\lim_{x\to 0} ln\, x \to -\infty.$$

The objective here is to demonstrate that the trajectory of $ln(0)$ towards $-\infty$ goes through the complex plane. The proof of this is carried out by extending the definition for $n!$, which takes on integer values $\forall n \geq 1$, to $\forall n \geq -\infty$, from which we demonstrate that $ln(0)$ is a complex number.

Proof

We begin with the definition of n!:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 \qquad \forall n \geq 1 \tag{1}$$

We also show below that:

$$\frac{n!}{(n-m)!} = n \times (n-1) \times \cdots \times (n-m+1) \quad \text{for integer } n \geq m \tag{2a}$$

Equating m in 2a to n leads to:

$$\frac{n!}{0!} = n \times (n-1) \times \cdots \times 2 \times 1 \tag{2b}$$

which, upon comparing with Equation 1, implies that:

$$0! = 1 \tag{2c}$$

Combining Equations 1 and 2b yields:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 \times 0! \tag{3}$$

Conjecture: Generalise $n!$ by expressing 0! in Equation 1 as:

$$0! = 1 = 0 \times (-1) \times (-2) \ldots \times -\infty \tag{4a}$$

allowing $n!$ to extend from integer n going all the way down to $-\infty$, i.e:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 \times 0 \times -1 \times -2 \times \ldots \times -\infty \tag{4b}$$

while still satisfying Equations 2 and 3.

Writing Equation 4a as:

$$1 = 0 \times \prod_{j=1}^{\infty} (-j) \tag{4c}$$

and substituting the identity $e^{i\pi} = -1$, where $i = \sqrt{-1}$, into 4b yields:

$$1 = 0 \times \prod_{j=1}^{\infty} (e^{i\pi} j) \tag{5}$$

Taking the natural logarithm of both sides of 5 and re-arranging leads to:

$$ln(0) = -\sum_{j=1}^{\infty} \{ln(j) + i\pi\}$$

or

$$ln(0) = -\sum_{j=1}^{\infty} ln(j) - \lim_{j\to\infty} \{ij\pi\} \tag{6}$$

which can be re-written as:

$$ln(0) = -\lim_{j\to\infty} \{ln(j!) + ij\pi\} \tag{7}$$

Using Stirling approximation for $j!$ in the limit $j \gg 1$ (Bender and Orszag, 1999):

$$\lim_{j>>1} j! \sim \sqrt{2\pi j}\, \left\{\frac{j}{e}\right\}^{j} \tag{8a}$$

or:

$$\lim_{j\gg 1} ln(j!) \sim \lim_{j\gg 1} ln \left\{\sqrt{2\pi j}\left\{\frac{j}{e}\right\}^{j}\right\} = \lim_{j\gg 1} \left\{\frac{1}{2}\ln(2\pi j) + jln(j) - j\right\} \quad (8b)$$

Substituting 8b into 7 gives:

$$ln(0) = -\lim_{j\to\infty} \left\{\frac{1}{2}\ln(2\pi j) + jln(j) - j + ij\pi\right\} \quad (9)$$

Note that Equation 9 has a real component, x, and an imaginary component, y, where:

$$x \equiv -\lim_{j\to\infty} \left\{\frac{1}{2}\ln(2\pi j) + jln(j) - j\right\} \quad (10a)$$

and:

$$y \equiv -\lim_{j\to\infty} \{j\pi\} \quad (10b)$$

Given that, in the limit $j \to \infty$, x in Equation 10a has a dominant term, $jln(j)$, such that $jln(j) \gg \frac{1}{2}\ln(2\pi j) - j$, we re-write Equation 9 as:

$$ln(0) = -\lim_{j\to\infty} \{jln(j)\} - i\lim_{j\to\infty} \{j\pi\} \quad (11)$$

Equation 11, therefore, shows that $ln(0)$ is a complex number, consisting of both a real and an imaginary component, i.e. $\lim_{j\to\infty} \{-jln(j)\}$ and $\lim_{j\to\infty} \{-j\pi\}$, respectively.

VISUALISING IN THE COMPLEX PLANE

With x and y, as given Equations 10a-b, both negative and representing the complex number z, with $z = x + iy$, it is noted that the point falls in the third quadrant of the complex plane, as illustrated in Figure 1. Figure 2 displays the path of $\arg(z)$ and θ as $j \to \infty$, where:

$$\arg(z) \equiv \sqrt{x^2 + y^2} \quad (12a)$$

and

$$\theta \equiv tan^{-1}\left[\frac{y}{x}\right] \quad (12b)$$

with x and y above coming from Equations 10a and 10b, respectively.

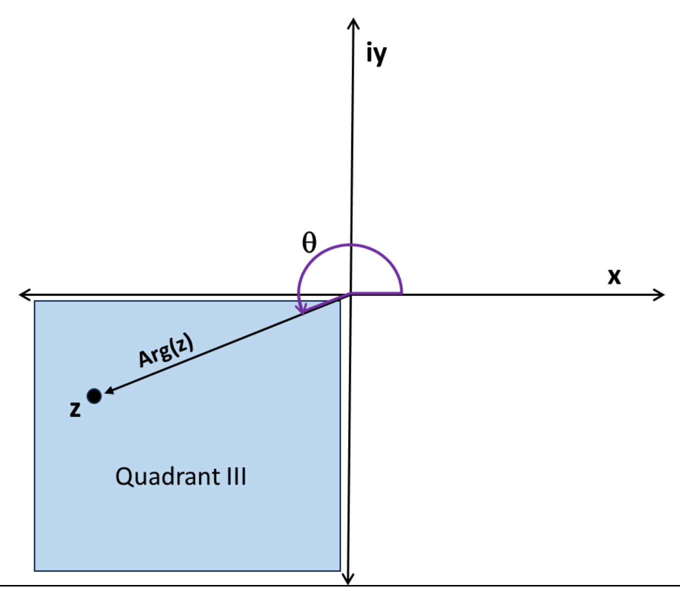

Figure 1 – Schematic of $z = x + iy$ in the complex plane.

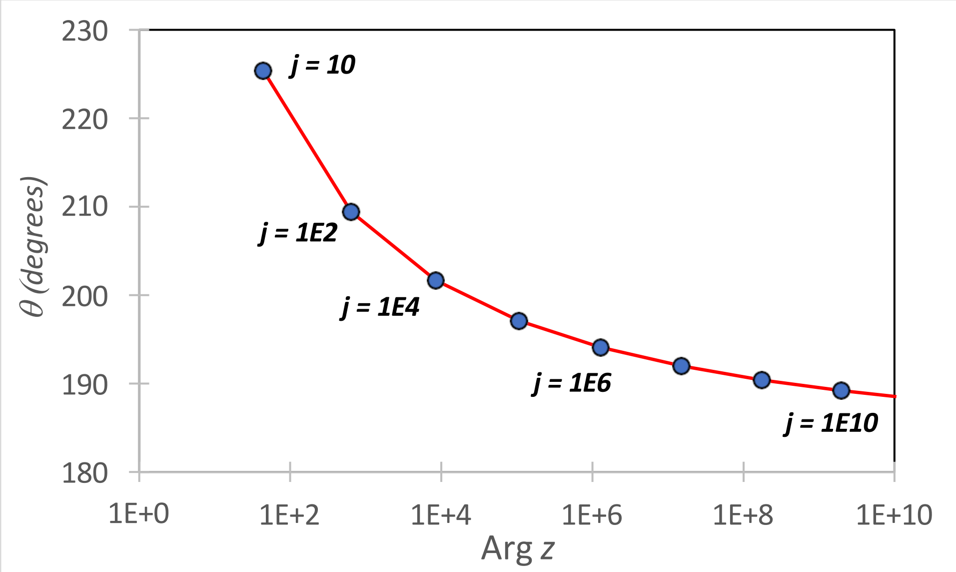

Figure 2 – The trajectory of Equations 10a and 10b with growing j. It is noted that, as $j \to \infty$, $Arg\, z \to \infty$ and $\theta \to \pi^-$ (or 180 in the third quadrant of the z-plane).

REFERENCES

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