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PARAMETER-UNIFORM HYBRID NUMERICAL SCHEME FOR SINGULARLY PERTURBED PARABOLIC CONVECTION-DIFFUSION PROBLEMS WITH DISCONTINUOUS INITIAL CONDITIONS

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ABSTRACT
This article presents a parameter-uniform hybrid numerical technique for singularly perturbed parabolic convection-diffusion problems (SPPCDP) with discontinuous initial conditions (DIC). It utilizes the classical backward-Euler technique for time discretization and a hybrid finite difference scheme (which is a proper combination of the midpoint upwind scheme in the outer regions and the classical central difference scheme in the interior layer regions(generated by the DIC)) for spatial discretization. The scheme produces parameter-uniform numerical approximations on a piecewise-uniform Shishkin mesh. When the perturbation parameter $\varepsilon(0 < \varepsilon \ll 1)$ is small, it becomes difficult to solve these problems using the classical numerical methods (standard central difference or a standard upwind scheme) on uniform meshes because discontinuous initial conditions frequently appear in the solutions of this class of problems. The method is shown to converge uniformly in the discrete supremum with nearly second-order spatial accuracy. The suggested method is subjected to a stability study, and parameter-uniform error estimates are generated. In order to support the theoretical findings, numerical results are presented.

Keywords: *Singularly perturbed parabolic convection-diffusion problem, finite difference scheme, Discontinuous initial condition, Interior and Boundary Layer, Piecewise-uniform S-type mesh.*

Subject Classifications: AMS 65M06; 65M12; 65M15

INTRODUCTION

This article investigates singularly perturbed parabolic partial differential equations (SPPDE) with interior layers, which are caused by the discontinuity in the initial condition (DIC) [25, 16, 1, 7, 12, 14, 15, 31]. SPPDE appears in many areas of science and engineering, including the simulation of oil extraction from underground reservoirs, fluid flows such as water quality problems in river net- works, convective heat transport problems with large Péclet numbers, and so on [9, 8]. Singularly perturbed problems (SPPs) are convection-diffusion processesresulting from convection-diffusion linearization and are used in fields such as oil extraction, fluid flows, and convective heat transport problems. These problems are described by partial differential equations (PDEs) with the highest spatial derivative multiplied by an arbitrarily small parameter ε [9, 20, 27]. In general, the solu- tions of these problems possess boundary layers which are thin regions in the neighborhood of the boundary of the domain, where the gradient of the solution steepens as the perturbation parameter ε tends to zero. Unless the mesh size is reduced in comparison to the diffusion parameter, i.e., the singular perturbation parameter, classical numerical methods on uniform meshes may fail to produce good numerical approximate solutions to these issues. The domain must be discretized using layer-adapted nonuniform meshes to get parameter-uniformly convergent numerical solutions to SPPs using classical finite difference techniques [32, 21, 19]. Because of these problems, numer- ical solutions to SPPs have grown in popularity among applied mathematicians and engineers. To get uniformly convergent numerical solutions of SPPs, numerous approaches are available in the literature; for more information, see the books by Miller et al. ([20]), Farrell et al. [9], Roos et al. [27], and Shishkin and Shishkina [29].

The main purpose of this article is to design and analyze a parameter-uniform hybrid scheme for solving SPPCDP with DIC. The parameter-uniform hybrid scheme for the SPPCDP with DIC is devised, produced, evaluated, and examined. We are interested in developing a parameter- uniform numerical technique [20] for this class of SPP in this study. SPPs cannot be solved effectively using ordinary numerical methods on uniform meshes due to the presence of boundary and interior layers in their solutions. Developing parameter-uniform numerical methods is thus a well-established principle in the study of numerical solutions to SPPs. We propose a higher-order uniformly convergent numerical scheme on the layer-adapted meshes like the piecewise-uniform Shishkin for singularly perturbed convection-diffusion problems with DIC and source term. Many ε-uniform numerical methods for stationary and non-stationary problems have been developed over the last few decades by many researchers; see the books [17, 6, 22, 1]. Roos et al. [27, 9] discussed numerical techniques for singularly perturbed differential equations. Natesan and Mukherjee [22, 21, 26], a parameter-uniform hybrid numerical scheme for the time-dependent convection-dominated initial boundary value problems. Clavero et al. [6, 4] and Cai and Liu [7, 3] have developed a variety of parameter uniform convergent methods based on Shishkin mesh. O'Riordan and Shishkin [29, 28, 24] considered parabolic problems with discontinuity as a function of time and developed a parameter-uniform numerical scheme using an implicit upwind difference scheme. Cen [2] developed an almost second-order parameter-uniform approximation using a hybrid difference scheme on the Shishkin mesh.

Mukherjee and Natesan [22, 21], discussed ε-Uniform error estimate of hybrid numerical scheme for SPPCDPs with interior layers. Mukherjee and Natesan [23, 22] considered a singularly perturbed parabolic convection-diffusion one-dimensional problem and the proposed numerical scheme consists of a classical backward-Euler method for the time discretization and a hybrid finite difference scheme for the spatial discretization, which proved second-order spatial accurate. According to Shishkin et al. [29, 30], parameter-uniform numerical methods for singularly perturbed parabolic problems with a DIC have been analyzed and examined. Rather than using hybrid (combination of midpoint upwind and central) finite difference operators on a piecewise-uniform mesh, Shishkin [6, 23] uses suitable fitted operator methods to capture the singularity in the neighborhood of the discontinuity. Gracia and E. O'Riordan [12, 11], discussed parameter-uniform approximations for a SPPCDP with a DIC. Gracia and E. O'Riordan [33, 10, 11], they construct and design numerical approxi- mations to a singularly perturbed problem with a DIC. Clavero et al.[3, 6, 5], proved an efficient numerical scheme for 1D parabolic SPP with interior and boundary layers. Mukherjee and Nate- san, [22, 23], presented time-dependent convection-dominated initial-boundary-value issues with parameter-uniform hybrid numerical techniques. Roos and Linns [27] characterized such meshes for the first time with the help of the mesh-generating functions and utilizing it, they easily deduced ε- uniform converges of the simple upwinding and linear finite elements applied to singularly perturbed boundary value problem exhibiting regular boundary layer. Recently, for SPPCDP, Mukherjee and Natesan [22], improved a hybrid numerical approach for SPP with parameter-uniform interior lay- ers. In recent years, regarding SPP with discontinuous convection coefficients possessing strong interior layers, Hemker et al. [15, 16] analyzed ε-uniform convergence of the standard upwind scheme for a stationary case. This paper addresses the SPPCDP with a DIC and focuses on the position of the interior layer function (2.1). When the parameter is extremely small, an interior layer follows a characteristic curve associated with the reduced problem. In a related work by [6], we examined a similar SPCDP with a DIC set by $a = 0$. The paper identifies an analytical function that aligns with the DIC while satisfying a differential equation with constant coefficients. The position of the interior layer function changes over time in the corresponding convection-diffusion problem, and numerical methods must track this position. Techniques like the Shishkin mesh tackle the problem for initial conditions $\phi \in C^0(0, 1)\backslash C^1(0, 1)$. The paper assumes that the convective coefficient is smooth, strictly positive, and solely time-dependent. An explicit discontinuous func- tion, denoted as $\tilde{S}(x, t)$, captures the singularity associated with the DIC. Asymptotic expansions for the solution $u(x, t)$, including this singularity, are constructed. When subtracting this singular function, the remaining $\tilde{y} = u(x, t) - \tilde{S}(x, t)$ becomes the solution of an SPCDP.

With this motivation, the objective is to develop and analyze a parameter-uniform hybrid numerical approach for solving the SPPCDP with DIC (2.1). The paper proposes a numerical scheme that combines the backward-Euler method for time discretization and a hybrid finite differ- ence scheme for spatial discretization. This hybrid scheme integrates the classical backward-Euler method for time discretization with a hybrid finite difference scheme for spatial discretization. We also analyze the truncation error associated with our proposed approach, perform stability analysis, and present numerical examples to validate the theoretical findings.

The remainder of the paper is organized as follows: Section 2, we define the continuous problem to be examined, define the singular function $\tilde{S}(x, t)$associated with DIC, and present prior bounds on the derivatives of the remainder term $\tilde{y}(x, t)$. Section 3, introduces a numerical scheme in the transformed domain based on a hybrid scheme. Section 4, discussed uniformly convergent of fully discrete scheme. Section 5, discusses error analysis and their stability. Section 6, includes numerical examples to validate the theoretical results. The paper concludes with the conclusions.

Notations Domains are represented by $\Omega = (0, 1)$, $\tilde{d} = \tilde{d}(0)$, $D = \Omega \times (0, T)$. In this paper, we use C as a constant independent of both the singular perturbation parameter ε and all discretization parameters.

A function's jump (a jump of a function) ϕ at a discontinuity point $\tilde{d}$ is also defined by

$$[\phi](\tilde{d}) = \phi(\tilde{d}^+) - \phi(\tilde{d}^-).$$

The L_∞ norm on the domain D will be denoted by $||.||_D$. We also denote the following interior layer function

$$E_\gamma(x,t) = \exp(-\frac{\gamma\left(x-\tilde{d}(t)\right)^2}{4\varepsilon t}), \quad 0 < \gamma \leq 1.$$

The finite difference operators L, L, L_ε, and $L^{N,M}$ denotes to approximate the differential operator L and will be discussed further.

We denote $||.||_{\overline{D}}$ is the maximum norm over any region D, which is defined by

$$||g||_{\overline{D}} = \max_{x \in \overline{D}} |g(x)|$$

for any function g.
The space $C^{0+\gamma}(D)$ where $D \subset R^2$, denotes an open set and is defined by

$$C^{n+\gamma}(\mathrm{D}) = \left\{z : \frac{\partial^{i+j} z}{\partial x^i \partial y^j} \in C^{0+\gamma}(\mathrm{D}), \quad 0 \leq i+2j \leq n\right\}, \quad \text{for each integer} \quad n \geq 0,$$

$C^{0+\gamma}(D)$ represents Hölder continuous functions of degree $0 < \gamma < 1$.

THE CONTINUOUS PROBLEM

We consider the following 1D SPPCDP with DIC on domain D:

$$\begin{cases} \mathcal{L}u(x,t) = u_t(x,t) - \varepsilon u_{xx}(x,t) + a(x,t)u_x(x,t) + b(x,t)u(x,t) = f(x,t), \quad (x,t) \in D, \\ u(x,0) = \phi(x), 0 \leq x \leq 1; \quad [\phi](\tilde{d}) \neq 0, 0 < \tilde{d} <= \mathcal{O}(1) < 1; \\ u(0,t) = u(1,t) = 0, \quad 0 < t \leq T; \\ a(x,t) > \alpha > 0, \quad \forall (x,t) \in D, \quad 0 \leq t \leq T, \quad a, f \in C^{4+\gamma}(\bar{D}), \quad \gamma > 0, \\ \phi^{(i)} \in C^4(((0,1)) \setminus \left\{\tilde{d}\right\}) : \phi^{(i)}(0) = \phi^{(i)}(1) = 0; \quad 0 \leq i \leq 4. \\ f^{(i+2j)}(p,0) = 0; \quad 0 \leq i+2j \leq 4-2p, \quad p = 0,1; \\ a_x(\tilde{d},0) = 0, \quad [\phi'](\tilde{d}) = 0. \end{cases} \tag{2.1}$$

where $D = \Omega \times (0, T]$, $\Omega = (0, 1)$, $t \in (0, T]$, $T > 0$. Here ε is a perturbation parameter such that $0 < \varepsilon \ll 1$ and the coefficients $a(x, t)$, $b(x, t)$ are smooth functions satisfying the followings:

$$a(x, t) \geq \alpha > 0, \; b(x, t) \geq \beta \geq 0 \text{ on } \overline{\Omega}.$$

Moreover, $[\phi]$ denotes the jump in the function ϕ across the point of discontinuity $x = \tilde{d}$, that is, $[\phi](\tilde{d}) = \phi(\tilde{d}^+) - \phi(\tilde{d}^-)$. In general, due to the presence of a discontinuity in the convection coefficient $a(x, t)$, the solution $u(x, t)$ of the problem (2.1) possesses an interior layer in the neighborhood of the point $x = \tilde{d}$. We observe that the initial function $\phi(x)$ is discontinuous at $x = \tilde{d}$ and the location of this point does not depend on the singular perturbation parameter ε. We assume that the initial data ϕ are sufficiently smooth functions on the domain $\bar{D}$ [?] and that satisfy sufficient compatibility conditions at the corner points (0, 0) and (1, 0).

We also assume that the required compatibility conditions at the transition point $(\tilde{d}, 0)$ follow a similar pattern. Assuming sufficient smoothness and compatibility conditions on u_0 and f, the parabolic problem (2.1) typically has a unique solution $u(x, t)$. This solution displays a regular boundary layer of width $O(\varepsilon)$ at $x = 1$. Additionally, in the range $a(t) > \alpha > 0, 0 \le t \le T, a, f \in C^4 + \gamma(\bar{D})$, we presume that b and f constitute suitably regular layer components. Moreover, we assume adequate compatibility at the points (0, 0) and (1, 0) to ensure $u \in C^{4+\gamma}(\bar{D})$.

Let there be a point $\tilde{d} \in (0, 1)$ such that ϕ is not continuous at $x = \tilde{d}$, but $\phi \in C4(\bar{\Omega} \setminus \{\bar{d}\})$

Since $a > 0$, the function $\tilde{d}(t)$ is monotonically increasing. Therefore we assume that the convection term $a(x, t)$ depends on both time and space variables. Then the location of the interior layer does not remain at the same position throughout the process. Thus, we need to track the movement of the layer. The path of the characteristic curve Γ is defined by the following:

$$\Gamma = \left\{ \left(\tilde{d}(t), t\right) : \tilde{d}'(t) = a(\tilde{d}(t), t), \quad \tilde{d}(0) = \tilde{d}, \quad 0 < d < 1 \right\} \tag{2.2}$$

We note that the characteristic curve Γ is generally not a straight line. Since $a(x, t) > 0$, the curve Γ is strictly increasing. Therefore, we have to restrict the final time T in order to avoid the overlap (to extend over or past around and cover) between the interior layer and the boundary layer regions. We also restrict the size of the final time T so that the interior layer does not interact with the boundary layer. We note that Gracia and E. O'Riordan [12, 14] proved that the restriction can be defined by the following relation:

$$0 < \frac{1 - \tilde{d}(T)}{1 - \tilde{d}} = \delta < 1, \quad \text{where } \tilde{\mathrm{d}}(\mathrm{T}) \le 1 - \delta. \tag{2.3}$$

Next, we decompose the solution u of problem (2.1) into the following way:

$$u(x, t) = \frac{[\phi](\tilde{d})}{2} \tilde{S}(x - d, t) + \tilde{y}(x, t), \text{where} \quad [\phi](\mathrm{d}) = \phi(\tilde{\mathrm{d}}^{+}) - \phi(\tilde{\mathrm{d}}^{-}) \tag{2.4}$$

The discontinuity in the initial condition generates an interior layer emanating from the point $(\tilde{d}, 0)$.

By identifying the leading term $_2\,[\phi](d)\psi_0$ in an asymptotic expansion of the solution, we can define the following continuous function

$$\tilde{y}(x, t) = u(x, t) - \tilde{S}(x, t) \tag{2.5}$$

where $\tilde{S}(x, t) = \frac{1}{2}[\phi](\tilde{d})\psi_0(x, t)$, $\psi_0(x, t) = erfc\left(\frac{\tilde{d}(t) - x}{2\sqrt{\varepsilon t}}\right)$, $erfc(z) = \frac{2}{\sqrt{\pi}} \int_{r=z}^{\infty} e^{-r^2} dr$ with

$$\mathcal{L}\tilde{y} = f + \frac{1}{2}[\psi](\tilde{d}) \left(a(\tilde{d}(t) - a(x)\right) \frac{\partial}{\partial s}\psi_0(x, t), \tag{2.6}$$

The component of function $\tilde{y}$ satisfies the following problem:

$$\begin{cases} \mathcal{L}\tilde{y} = 0, \quad (x, t) \in D, \\ \tilde{y}(0, t) = -\frac{1}{2}[\phi](\tilde{d})\psi_0(0, t), \quad [0, T], \\ \tilde{y}(1, t) = -\frac{1}{2}[\phi](\tilde{d})\psi_0(1, t). \quad [0, T], \quad \tilde{y}(x, 0) = \begin{cases} \phi(x), & x \le \tilde{d}, \\ \phi(\tilde{d}^{-1}), & x = \tilde{d}, \\ \phi(x) - [\phi](d), & x > \tilde{d} \end{cases} \end{cases} \tag{2.7}$$

Using the definition of the discontinuous function $\tilde{S}$, we observe the following:

$$\tilde{S}(x - \tilde{d}, 0) = \begin{cases} -1, x < \tilde{d}, \\ 0, \quad x = \tilde{d}, \\ 1, \quad x > \tilde{d}. \end{cases} \tag{2.8}$$

2.1 Transformation to fix the location of the interior layer

We observe that the in-homogeneous term in (2.6) is continuous, but not in $C^1(\bar{D})$ on the closed domain. The presence of an inhomogeneous term will induce an interior layer into the function y. So if the convective term $a(x, t)$ depends on the space variable, we are required to transform the problem (2.1) so that the curve Γ is transformed to a straight line, around which a layer-adapted mesh-like piecewise-uniform Shishkin mesh is constructed.

One possible choice for the transformation $X : (x, t) \longrightarrow (k, t)$ is the piecewise linear map given by [10, 12]

$$k(x,t) = \begin{cases} \frac{\tilde{d}}{\tilde{d}(t)}x, & x \leq \tilde{d}(t), \\ 1 - \frac{1-\tilde{d}}{1-\tilde{d}(t)}(1-x), & x \geq \tilde{d}(t), \end{cases} \tag{2.9}$$

which means that $a(\tilde{d}(t), t) = a(\tilde{d}, t)$. We note that $x = k$ at $t = 0$ and $x = d$ for all t such that $x = \tilde{d}(t)$. We also define two subdomains of D on either side of (left and right subdomains) Γ to be

$$D^- = \bar{\Omega}^- \times (0, T] = (0, \tilde{d}) \times (0, T] \qquad \text{and} \qquad D^+ = \bar{\Omega}^+ \times (0, T] = (\tilde{d}, 1) \times (0, T].$$

Using this map, the differential equation (2.1) can be transformed into the following problem: Find y such that

$$\begin{cases} \mathcal{L}\tilde{y} = g\left(f + \frac{1}{2}[\phi](\tilde{d})\frac{\left(a(\tilde{d},t) - a(k,t)\right)}{\sqrt{\varepsilon\pi t}}\exp^{-}\frac{g(k,t)(k-\tilde{d})^2}{4\varepsilon t}\right) + \\ \frac{1}{2}[\phi](d)\left(b(\tilde{d},t) - b(x,t)\right)g(x,t)\psi(x,t), \quad x \neq \tilde{d}, \\ [\tilde{y}]\left(\tilde{d},t\right) = 0, \quad \left[\frac{1}{\sqrt{g}}\tilde{y}_x\right]\left(\tilde{d},t\right) = 0, \end{cases} \tag{2.10}$$

with the following transformed initial conditions,

$$\tilde{y}(p,t) = -\frac{1}{2}[\phi](\tilde{d})\psi_0(p,t), \quad p \in \{0,1\}, \quad 0 \leq t \leq T, \tag{2.11}$$

$$\tilde{y}(x,0) = \begin{cases} \phi(x), & x < \tilde{d}, \\ \phi(\tilde{d}^-), & x = \tilde{d}, \\ \phi(x) - [\phi](\tilde{d}), & x > \tilde{d}. \end{cases} \tag{2.12}$$

Here, $\mathcal{L}\tilde{y} = -\varepsilon\tilde{y}_{xx} + k(x,t)\tilde{y}_x + g(x,t)(b(x,t)\tilde{y} + \tilde{y}_t)$ and the functions g, k are defined by,

$$k(x,t) = \left\{ \sqrt{g}\left(a(x,t) + a(\tilde{d},t)(\psi_d(x) - 1)\right), \right. \tag{2.13}$$

$$g(x,t) = \begin{cases} \left(\frac{d(t)}{\tilde{d}}\right)^2, & x < \tilde{d}, \\ \left(\frac{1-\tilde{d}(t)}{1-\tilde{d}}\right)^2, & x > \tilde{d}. \end{cases} \tag{2.14}$$

and

$$\psi_{\tilde{d}}(x) = \begin{cases} \frac{\tilde{d}-x}{\tilde{d}}, & x < \tilde{d}, \\ \frac{x-\tilde{d}}{1-\tilde{d}}, & x > \tilde{d}. \end{cases} \tag{2.15}$$

2.2 Bounds for the solution of continuous problem

In this section, we analyze the solution of the SPPCDPs defined by (2.1) and its derivatives. The solution's existence and uniqueness depend on the smoothness of $\phi(x)$ and the compatibility condition at the corner points, as described below:

$$\phi_0(0) = \phi_1(1) = 0, \tag{2.16}$$

and

$$\begin{cases} \frac{\partial\phi_1(0)}{\partial t} = -\varepsilon\frac{\partial^2\phi(0)}{\partial x^2} + a(0)\frac{\partial\phi(0)}{\partial x} + b(0)\phi(0) = f(0,0), \\ \frac{\partial\phi_2(1)}{\partial t} = -\varepsilon\frac{\partial^2\phi(1)}{\partial x^2} + a(1)\frac{\partial\phi(1)}{\partial x} + b(1)\phi(1) = f(1,0). \end{cases} \tag{2.17}$$

2.2.1. Decomposition of the solution

The sections that follow discuss solution decomposition. To develop sharp bounds in the error analysis, the SPPCDP with DIC (2.1) solution $u(x)$ is decomposed into $u(x, t) = v(x, t) + w(x, t) + z(x, t)$, $(x, t) \in \bar{D}$, where regular (smooth) component $p(x, t)$, singular component $q(x, t)$, and inte- rior layer component $z(x, t)$. The smooth component $p(x, t)$ can be expressed using an asymptotic expansion:

$$v(x,t) = \sum_{i=0}^{4} \varepsilon^i v_i(x,t) = p_0(x,t) + \varepsilon v_1(x,t) + \varepsilon^2 v_2(x,t) + \varepsilon^3 v_3(x,t) + \varepsilon^4 v_4(x,t), \quad (x,t) \in \bar{D}.$$

where the functions $v(x,t)$ satisfies the following equations

$$\begin{cases} \partial \dfrac{\partial v_0}{\partial t} + a\dfrac{\partial v_0}{\partial x} + bv_0 = f, \quad \text{in} \quad D^- \cup D^+ \\ v_0(0,t) = u(0,t), \quad t \in (0,T], \quad v_0(x,0) = u(x,0), \quad x \in \bar{\Omega}, \\ \partial \dfrac{\partial v_1}{\partial t} + a\dfrac{\partial v_1}{\partial x} + bv_1 = -\dfrac{\partial^2 v_0}{\partial x^2}, \\ v_1(0,t) = 0, \quad v_1(1,t) = 0, \quad t \in (0,T], \quad v_1(x,0) = 0, \quad x \in \bar{\Omega} \\ \partial \dfrac{\partial v_2}{\partial t} + a\dfrac{\partial v_2}{\partial x} + bv_2 = -\dfrac{\partial^2 v_1}{\partial x^2}, \\ v_2(0,t) = 0, \quad t \in (0,T], \quad v_2(x,0) = 0, \quad x \in \bar{\Omega}, \end{cases} \tag{2.18}$$

and lastly, the functions v_3 satisfies

$$\begin{cases} \mathcal{L}v_3 = -\dfrac{\partial^2 v_3}{\partial x^2}, \quad \text{in} \quad D \\ v_3(0,t) = v_3(1,t) = 0, \quad t \in (0,T], \quad v_3(x,0) = 0, \quad x \in \bar{\Omega}. \end{cases} \tag{2.19}$$

Hence, the smooth component of the solution satisfies the discontinuous function $v(x,t)$ by

$$\begin{cases} \mathcal{L}v = f, \quad (x,t) \in D^- \cup D^+, \\ v(x,0) = u(x,0), \quad x \in \bar{\Omega}, \\ v(0,t) = u(0,t), \quad v(\tilde{d}^-,t) = g_1(t, \\ v(\tilde{d}^+,t) = g_2(t), \quad v(1,t) = u(1,t), \quad t \in (0,T], \end{cases} \tag{2.20}$$

where $v(\tilde{d}^{\pm}, t) = \lim_{x \to \tilde{d} \pm 0} v(x, t)$ and the suitable choices of the functions $g_1(t)$ and $g_2(t)$ will be obtained from Theorem 2.2.

We define the discontinuous function w, which represents the singular component of the decompo- sition, as follows:

$$\begin{cases} \mathcal{L}w = 0, \quad (x,t) \in D^- \cup D^+, \\ w(0,t) = w(1,t) = 0, \quad w(x,0) = 0, \\ [w](\tilde{d},t) = -[v](\tilde{d},t), \quad [(w)_x](\tilde{d},t) = -[(v)_x](\tilde{d},t), \\ \text{Hence} \quad w(\tilde{d}^-,t) = u(\tilde{d}^-,t) - v(\tilde{d}^-,t) \quad \text{and} \\ w(\tilde{d}^+,t) = u(\tilde{d}^+,t) - v(\tilde{d}^+,t). \end{cases} \tag{2.21}$$

We define the discontinuous function z, which constitutes the interior component of the decompo- sition, as follows:

$$\begin{cases} \mathcal{L}z = 0, \quad (x,t) \in D^- \cup D^+, \\ [z](\tilde{d},t) = -[v](\tilde{d},t), \quad [z_x](\tilde{d},t) = -[v_x + w_x](\tilde{d},t), t \geq 0, \\ z(0,t) = z(1,t) = 0, \quad t \geq 0, \quad z(x,0) = 0, \quad x \in \Omega^-, \\ z(x,0) = \phi(x,0) - w(x,0), \quad x \in \Omega^+. \end{cases} \tag{2.22}$$

Theorem 2.1. *Assume in equations (2.1) that the data a,* $b \in C^2(\bar{\Omega})$*,* $f \in C^{2+\lambda}(\bar{D})$ *and the con- vection coefficient* $a(x, t) \geq \alpha > 0$*,* $x \in \bar{\Omega}$ *.Also, let the initial data* ϕ *be identically zero so that the compatibility conditions*

$$\frac{\partial^{i+k} f}{\partial x^i \partial t^j}(0,0) = \frac{\partial^{i+k} f}{\partial x^i \partial t^j}(1,0) = 0, \quad for \quad 0 \leq i + 2k \leq 2,$$

be fulfilled. Then the solution $u(x,t)$ of the (2.1) lies in $f \in C^{4+\lambda}(\bar{D})$ and satisfies the estimates

$$\left\| \frac{\partial^{i+k} u}{\partial x^i \partial t^j}(x,t) \right\|_{D^- \cup D^+} \leq C\varepsilon^{-1}, \quad 0 \leq i + 2k \leq 4. \tag{2.23}$$

Theorem 2.2. *For all non-negative integers* $i, k, k \in N \cup \{0\}$ *satisfying* $0 \leq i \leq 3$ *and* $0 \leq i + 2k \leq 4$*,there exist smooth functions* $g_1(t), g_2(t)$ *such that the smooth component p given in (2.20) satisfying the bounds*

$$\left\| \frac{\partial^{i+k} v}{\partial x^i \partial t^j}(x,t) \right\|_{D^- \cup D^+} \leq C, \quad 0 \leq i + 2k \leq 4$$

$$\left\| \frac{\partial^4 v}{\partial x^4}(x,t) \right\| \leq C\varepsilon^{-1}, \quad (x,t) \in D^- \cup D^+,$$

and the boundary layer component q given in (2.21) satisfies the bounds

$$\left| \frac{\partial^{i+k} w}{\partial x^i \partial t^k}(x,t) \right| \leq \begin{cases} C\left(\varepsilon^{-i} e^{-(\tilde{d}-x)\alpha_1/\varepsilon}\right), & (x,t) \in D^-, \\ C\left(\varepsilon^{-i} e^{-(\tilde{d}-x)\alpha_2/\varepsilon}\right), & (x,t) \in D^+, \end{cases}$$

$$\left| \frac{\partial^4 w}{\partial x^4}(x,t) \right| \leq \begin{cases} C\left(\varepsilon^{-4} e^{-(\tilde{d}-x)\alpha_1/\varepsilon}\right), & (x,t) \in D^-, \\ C\left(\varepsilon^{-4} e^{-(\tilde{d}-x)\alpha_2/\varepsilon}\right), & (x,t) \in D^+. \end{cases}$$

where C is a constant independent of ε.

Theorem 2.3. *There exists a function r(t) such that the problem solutions v, w*

$$L_\varepsilon v = f, \quad ((x,t) \in D, \tag{2.24a}$$
$$v(x,0) = g_1(x), \quad v(-1,t) = u(-1,t),\ v(1,t) = r(t) \tag{2.24b}$$
$$L_\varepsilon w = 0, \quad (x,t) \in D, \tag{2.24c}$$
$$w(0,t) = 0,\ w(-1,t) = 0,\ w(1,t) = u(1,t) - r(t), \tag{2.24d}$$

such that v, w ∈ $C^{4+\gamma}(\bar{D})$ and the following regular component v satisfies the following bounds for $0 \leq i+2j \leq 4$

$$\left\|\frac{\partial^{i+j}}{\partial x^i \partial t^j} v^-(x,t)\right\|_{\bar{D}^-} \leq \left(1 + \varepsilon^{2-(i+j)}\right), \tag{2.25a}$$
$$\left\|\frac{\partial^{i+j}}{\partial x^i \partial t^j} v^+(x,t)\right\|_{\bar{D}^+} \leq C\left(1 + \varepsilon^{1-i/2}\right). \tag{2.25b}$$

Next, we decompose the boundary layer components w, for all $0 \leq i + 2j \leq 4$ and $(x, \bar{D})$, of the solution u of the discontinuous initial condition (2.1) to satisfy the following bounds:

$$\left|\frac{\partial^{i+j}}{\partial x^i \partial t^j} w_L(x,t)\right| \leq C\varepsilon^{-i/2} e^{-\frac{x}{\sqrt{\varepsilon}}}, \tag{2.26a}$$
$$\left|\frac{\partial^{i+j}}{\partial x^i \partial t^j} w_R(x,t)\right| \leq C\varepsilon^{-i/2)} e^{-\frac{1-x}{\sqrt{\varepsilon}}}. \tag{2.26b}$$

Theorem 2.4. *The interior layer component* $z \in C^{2+\gamma}(\overline{D}^-) \cup C^{2+\gamma}(\overline{D}^+)$ *satisfies the bounds*

$$|z(x,t)| \le Ce^{\dfrac{-\gamma g(x,t)(d-t)^2}{4\varepsilon t}}, \quad (x,t) \in D. \tag{2.27}$$

For $x \neq \tilde{d}$,

$$||\frac{\partial^{i+m} z}{\partial x^i \partial t^k}(x,t)|| \le C\left(1+\varepsilon^{-i/2}\right), \quad i+2m \le 3; \quad \text{and} \quad \left|\frac{\partial^2 z}{\partial t^2}(x,t)\right| \le C\left(1+\sqrt{\varepsilon/t}\right).$$

Proof. The detailed proof can be obtained by [10, 12].

Remark 2.5. *If* u *is the solution of the continuous problem (2.10) in the transformed domain and* U *be the solution of the discrete problem (3.1), then* $\hat{U}$ *is the solution obtained from* U *using the hybrid scheme. If* $[\phi](\tilde{d}) = 0$ *then the function* u *is decomposed as* $u = v + w + z$*. In this case, the constraint* $M = O(N)$ *can be removed by using the proof of the next theorem, and the following error estimate can be obtained:*

$$||\hat{U} - u||_{\overline{D}} \le CN^{-1}\ln N + CM^{-1}\ln M$$

If $[\phi'](\tilde{d}) = 0$ *then the error bound is dominated by the term* $CM^{-1/2}$*, which corresponds to the result in [9], that is,*

$$||\hat{U} - u||_{\overline{D}} \le CN^{-1}\ln N + C|[\phi'](d)|M^{-1/2}.$$

NUMERICAL SCHEME IN THE TRANSFORMED DOMAIN

In this section, we use the implicit Euler method for time discretization and a layer-adapted mesh of the Shishkin type for spatial discretization with a hybrid numerical scheme. The main focus is to investigate both the semi-discretization and spatial discretization of the model problem, which are crucial for analyzing the convergence of the fully discrete scheme.

We approximate the problem (2.7) using the implicit-Euler approach and typical central differences on layer-adapted meshes, such as the S-type mesh described in [19, 22, 32]. It's important to note that the interior layer, originating from $x = \tilde{d}$ and following the characteristic curve Γ, remains independent of the boundary layer near $x = 1$.

The Semidiscretization

This section discusses the temporal semi-discretization of the model problem (2.1) which is required for the fully discrete scheme's convergence analysis. We consider a uniform mesh to discretize the time domain $[0, T]$ and denote it with a uniform time step size Δt such that

$$\hat{\Omega}_t^M = \{t_n : t_n = n\Delta t,\ n = 0, \dots M,\ t_0 = 0, \quad t_M = T, \quad \Delta t = T/M\}$$

where M denotes the number of mesh intervals in the t-direction (temporal direction). By using the backward-Euler method to discretize the problem (2.1), we obtain the following semi discrete scheme:

$$\begin{cases} U^0 = \phi(x), \quad 0 \le x \le 1, \quad [\phi](\tilde{d}) \neq 0, \quad 0 < \tilde{d} < 1, \\ \begin{cases} (I + \Delta t \mathcal{L}_{x,\varepsilon})\hat{U}^{n+1} = U^{n+1}(x) + \Delta t f(x, t_{n+1}), \\ \left[\dfrac{D_x^0 U}{\sqrt{g}}\right](\tilde{d}, t_j) = 0, x_i = \tilde{d}, \quad t_j > 0, \\ \hat{U}^{n+1}(0,t) = \hat{U}^{n+1}(1,t) = 0, \quad t_j \ge 0 \\ \hat{U}^{N,\Delta t} = \hat{u}(x_i, t_j), \quad (x_i, t_j) \in \Gamma_\tau^{N,M}. \end{cases} \\ n = 0, 1, \dots, M-1, \end{cases} \tag{3.1}$$

where

$$\mathcal{L}_{x,\varepsilon} = u_t(x,t) - \varepsilon u_{xx}(x,t) + a(x,t)u_x(x,t) + b(x,t)u(x,t)$$

$$\left[\frac{D_x^0 U}{\sqrt{g}}\right](d, t_j) = \frac{1-d}{(1-d(t_j))} D_x^+ U(d, t_j) - \frac{d}{d(t_j)} D_x^- U(d, t_j).$$

D^0 is the central difference and I is the identity operator. In addition, $U^n(x)$ is the semi-discrete approximation to the exact solution $u(x, t)$ of the continuous problem (2.1) at t_n.

The operator $(I + \Delta t L_{x,\varepsilon})$ satisfies the following minimum principle, which ensures the stability of the scheme (3.1).

1. *Convergence Analysis*

Lemma *(Minimum Principle for the semi-discrete problem) Let D be any domain and $z \in C^2(\bar{D})$. If $z(x) \geq 0$ on the boundary of D and $(I + \Delta t L_{x,\varepsilon})\, z(x) \leq 0$, $\forall x \in D$, then $z(x) \geq 0$, $\forall x \in D$.*

Proof. The proof can be found in [21] and [22] for detailed information. In order to analyze the uniform convergence of the solution $\hat{U}^n(x)$ of (3.1) to the exact solution $u(x, t_n)$, we will do the stability analysis and also derive the consistency result of the scheme (3.1). It is clear that the operator $(I + \Delta t L_{x,\varepsilon})$ satisfies the following maximum principle:

$$|| \, (I + \Delta t \mathcal{L}_{x,\varepsilon})^{-1} \, ||_\infty \leq \frac{1}{1 + \beta \Delta t},$$

which ensures the stability of the scheme (3.1) and (see [21] for details).

Next, we define the local truncation error e_{n+1} for the semi-discrete scheme (3.1), and at the $(n + 1)$-th time level, this error is characterized by the following expression:

$$e_{n+1} = u(t_{n+1}) - \hat{u}^{n+1}, \ldots, n = 0, 1, \ldots, M - 1. \tag{3.2}$$

where $\hat{u}^{n+1}$ is the solution obtained after one step of the semi-discrete scheme, initialized with the exact value $u(t_n)$ instead of using $u^n(x)$ as the initial data. Specifically, we have:

$$\begin{cases} (I + \Delta t \mathcal{L}_{x,\varepsilon})\, \hat{u}^{n+1}(x) = -u(x, t_n) + \Delta t f(x, t_{n+1}), & x \in \Omega, \\ \hat{u}^{n+1}(0) = \hat{u}^{n+1}(1) = 0. \end{cases} \tag{3.3}$$

Lemma 3.2. *Let the local truncation error e_{n+1} satisfies*

$$||e_{n+1}||_\infty \leq (C\Delta t^2) \ .$$

Proof. One can refer [23].

The following lemma gives consistency results for the semidiscrete scheme:

Lemma 3.3. *Let u be the solution of the problem (2.1). If*

$$|\frac{\partial^l}{\partial t^l} u(x, t)| \leq C, \quad (x, t) \in \bar{\Omega} \times (0, T), \quad 0 \leq l \leq 2$$

then, the local error corresponding to the numerical scheme (2.1) satisfies

$$||e_{n+1}||_{\infty,\bar{\Omega}} \leq C((\Delta t)^2, \qquad n = 0, 1, \ldots, M - 1. \tag{3.4}$$

In order to show the uniform convergence of semidiscretizastion in (3.1), we introduce the global error E_n as

$$E_n = u(t_n) - u^n, \qquad 1 \leq n \leq M. \tag{3.5}$$

The following theorem demonstrates that the global error for time semidiscretization processes are first-order convergent.

Theorem 3.4. *The global error E_n in (3.5) satisfies*

$$\sup_{1 \leq n \leq M} ||E_n||_{\infty,\bar{\Omega}} \leq C\Delta t. \tag{3.6}$$

Proof: See [22] for details.

The Spatial discretization and the special meshes

2. *Construction of piecewise-uniform (Shishkin mesh)*

Assuming that N and $M = O(N)$ are both positive integers, we consider the domain $\bar{D} = \bar{\Omega} \times$

$[0, T] = [0, 1] \times [0, T]$. Additionally, let $N \geq 4$ be a positive even integer. We approximate solution of the problem (2.10) and (2.12) on a rectangular grid in the computation domain $\bar{D}^{N,M} = \{x_i\}^{N}_{i=0} \times \{x_j\}^{M}_{j=0}$ which concentrates mesh points in the interior and boundary layers with $\partial D^{N,M} = \bar{D}^{N,M} \setminus D$.

Since the SPPCDP with DIC (2.1) has a DIC has an interior layer at $x = \tilde{d}$, we construct a piecewise-uniform Shishkin mesh. The solution of the model problem (2.1) also exhibits boundary layers of width $O(\sqrt{\varepsilon})$ and interior layer emanating at a point $x = \tilde{d}$, we construct the rectangular mesh as follows. Based on the bounds (**??**) on the layer components, the Shishkin mesh is defined by using the following transition points.

We divide the transformed space domain [0, 1] into the following four sub-intervals as follows

$$[0, 1] = [0, \tilde{d} - \tau_1] \cup [\tilde{d} - \tau_1, \tilde{d} + \tau_2] \cup [\tilde{d} + \tau_2, 1 - \hat{\tau}_x] \cup [1 - \hat{\tau}_x, 1],$$

where the transition points τ_1, τ_2 and $\hat{\tau}_x$ are defined by

$$\begin{cases} \hat{\tau}_x = \min\left\{\frac{1}{4}, 2\sqrt{\varepsilon/\alpha}\ln N\right\}, \\ \tau_1 = \min\left\{\frac{\tilde{d} - \hat{\tau}_x}{4}, 2\sqrt{T\varepsilon}\ln N\right\}, \\ \tau_2 = \min\left\{1 - \tilde{d}(T), \frac{\tilde{d} - \hat{\tau}_x}{4}, 2\sqrt{T\varepsilon/\delta}\ln N\right\}. \end{cases} \tag{3.7}$$

On each subinterval, a uniform mesh with $N/4$ mesh intervals is placed such that

$$\hat{\Omega}_x^N = \left\{x_i : 0 \leq x \leq \tilde{d} - \tau_1\right\} \cup \left\{\tilde{d} + \tau_2 \leq x \leq 1 - \hat{\tau}_x\right\} \tag{3.8}$$

denotes the set of interior points of the mesh.

The mesh interval point N of spatial grids are distributed into four intervals in the ratio $\frac{3N}{8} : \frac{N}{4} : \frac{N}{4} : \frac{N}{8}$ and each of them is spaced uniformly.

Thus, the computational domain is defined as

$$\bar{D}^{N,M} = \hat{\Omega}_x^N \times \hat{\Omega}_t^M,$$

where

$$\hat{\Omega}_x^N = \{x_i : x_{i-1} + h_i, x_0 = 0, x_N = 1, 1 \leq i \leq N\}.$$

Then, obviously, $x_{N/2} = \tilde{d}$ and $\bar{\Omega}^N = \{x_i\}_{i=0}^N$ Let us denote the step sizes in space by $h_i = x_i - x_{i-1}$, $i = 1, \ldots, N$ and let $\hat{h} = h_i + h_{i+1}$, for $i = 1, \ldots, N-1$. Further, we denote the mesh size h_i in spatial direction as follows:

$$h_i = \begin{cases} H_{(l)} = \frac{4(\tilde{d} - \tau_1)}{N}, & \text{for } i = 1, \ldots, N/4, \\ h_{(l)} = \frac{4\tau_1}{N}, & \text{for } i = N/4 + 1, \ldots, N/2 - 1, \\ h_{(r)} = \frac{4\tau_2}{N}, & \text{for } i = N/2 + 1, \ldots, 3N/4, \\ H_{(r)} = \frac{4(1 - \tilde{d} - \tau_2)}{N}, & \text{for } i = 3N/4 + 1, \ldots, N. \end{cases} \tag{3.9}$$

3. *Hybrid finite difference scheme*

We propose a numerical scheme that combines the classical backward-Euler method for tempo- ral discretization with the mid-point upwind scheme in the outer regions and the classical central difference scheme in the interior layer regions and the boundary layer region for the spatial dis- cretization. [1] We define the difference operators like forward D_x^+, backward D_x^-, and central difference D^0 in space, second-order central difference operator δ^2, and backward difference operator D_t^- in time, respectively for the mesh functions $v(x_i, t_j)i = v^j$ on $\bar{D}^{N,\Delta t}$. We also define

$v^n_{j\pm1/2} = (v^n_j + v^n_{j\pm1})/2, \quad a_{j\pm1/2} = (a_j + a_{j\pm1})/2, \quad b^n_{j\pm1/2} = (b^n_j + b^n_{j\pm1})/2, \quad f^n_{j\pm1/2} = (g^n_j + f^n_{j\pm2})/2.$

We note that, for a given function $v(x_i, t_j) = v_{ij}$ then we similarly define $v_{i\pm1/2} = \dfrac{(v_{i\pm1} + v_i)}{2}$.

Now, we are ready to define the hybrid numerical scheme for the SPPCDP (2.1) which consists of a proper combination of the midpoint upwind scheme in the outer regions $[0, \tilde{d} - \tau_1]$, $[\tilde{d} - \tau_1, \tilde{d} + \tau_2]$ and the central difference scheme which is used in the interior layer regions $[\tilde{d} + \tau_2, 1 - \hat{\tau}_x]$ and boundary layer region $[1 - \hat{\tau}_x, 1]$ to discretize the spatial derivatives.
The difference operator of DIC is denoted by

$$\mathcal{L}^N_{dis} U^{n+1}_i = D^F_x U^{n+1}_i - D^B_x U^{n+1}_i.$$

where, $D^+_{dis,x}$ denotes the forward, $D^-_{dis,x}$ backward DIC.

The finite difference operator is denoted by L^N, and a hybrid numerical scheme is developed by combining the three schemes, which takes the form:

$$\begin{cases} \mathcal{L}^N_{cen} U^{n+1}_i = f^{n+1}_i, & for \quad i = 1, \ldots, N/4, N/2+1, \ldots, 3N/4, \\ \mathcal{L}^N_{mu} U^{n+1}_i = f^{n+1}_{i-\frac{1}{2}}, & for \quad i = N/4+1, \ldots, N/2, 3N/4+1, \ldots, N-1, \\ \mathcal{L}^N_{dis} U^{n+1}_i = D^F_x U^{n+1}_i - D^B U^{n+1}_i = 0, & for \quad i = \frac{N}{2}. \end{cases} \tag{3.10}$$

and

$$\begin{cases} D^F_x U^{n+1}_i = \left[-U^{n+1}_{N/2+2} + 4U^{n+1}_{N/2+1} - 3U^{n+1}_{N/2}\right]/2h_{(r)}, \\ D^B_x U^{n+1}_i = \left[U^{n+1}_{N/2-2} - 4U^{n+1}_{N/2-1} + 3U^{n+1}_{N/2}\right]/2h_{(l)}. \end{cases} \tag{3.11}$$

[1]Before describing the computational scheme, we define the following notations for various finite difference operators D^+, D^-, D^0, δ^2 and D^-:

$$aD_x v(x_i, t_j) = \frac{1}{2}(a(x_i, t_j))D^-_x v(x_i, t_j) + \frac{1}{2}(a(x_i, t_j) - |a(x_i, t_j)|)D^+_x v(x_i, t_j),$$

$$D^-_t v(x_i, t_j) = \frac{v(x_i, t_j) - v(x_i, t_{j-1})}{\Delta t},$$

$$D^-_x v(x_i, t_j) = \frac{v(x_i, t_j) - v(x_{i-1}, t_j)}{h_i},$$

$$D^+_x v(x_i, t_j) = \frac{v(x_{i+1}, t_j) - v(x_i, t_j)}{h_{i+1}},$$

$$D^0_x v(x_i, t_j) = \frac{v(x_i, t_{j+1}) - v(x_{i-1}, t_j)}{h_i},$$

$$\delta^2_x v(x_i, t_j) = \frac{2}{h_i + h_{i+1}}\left(D^+_x v(x_i, t_j) - D^-_x v(x_i, t_j)\right)$$

where

$$\begin{cases} \mathcal{L}^N_{cen} U^{n+1}_i = \varepsilon D^+_x D^-_x U^{n+1}_i + a_i D^-_x U^{n+1}_i + b_i - D^-_t U^{n+1}_i, & i = 1, \ldots, N/4, N/2+1, \ldots, 3N/4, \\ \mathcal{L}^N_{dis} U^{n+1}_i = \varepsilon D^+_{dis,x} D^-_{dis,x} U^{n+1}_i + a_i D^0_{dis,x} U^{n+1}_i + b_i U^{n+1}_i - D^-_{dis,t} U^{n+1}_i, & i = \frac{N}{2}, \\ \mathcal{L}^N_{mu} U^{n+1}_i = \varepsilon D^+_x D^-_x U^{n+1}_i + a_{i-\frac{1}{2}} D^-_x U^{n+1}_i + b_{i-\frac{1}{2}} - D^-_t U^{n+1}_{i+\frac{1}{2}}, & i = N/4+1, \ldots, N/2, 3N/4+1, \ldots, N-1, \end{cases} \tag{3.12}$$

and

$$f_i = f(x_i) = \begin{cases} \left[m_i f^{n+1}_{i-1} + m^0_i f^{n+1}_i + m^+_i f^{n+1}_{i+1}\right], & \text{for} \quad \mathrm{i} = 1, \ldots, \mathrm{N}/2, \\ 0, \quad \text{for} \quad \mathrm{i} = \mathrm{N}/2, \\ f_i, \quad i = N/2+1, \ldots, N-1, \end{cases} \tag{3.13}$$

After rearranging the terms in (3.10)-(3.12),we obtain the following complete hybrid scheme on the mesh $\bar{D}^{N,M}$:

$$\begin{cases} U_i^0 = \phi(x_i), \quad 0 \le i \le N, \quad [\phi](\tilde{d}) \neq 0, \quad 0 < \tilde{d} < 1, \\ \begin{cases} \mathcal{L}_{hyb}^N U_i^{n+1} \equiv \left[r_i^- U_{i-1}^{n+1/2} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1}\right] + \left[Q_i^- U_{i-1}^{n+1} + Q_i^0 U_i^{n+1} + Q_i^+ U_{i+1}^{n+1}\right] \\ = \hat{f}_i^{n+1}, \quad for \quad i = 1, \dots, N/2+1, \dots, N-1, \\ 0, \quad i = \dfrac{N}{2} \\ \mathcal{L}^* = 0, \quad x_i = \tilde{d}, \quad t_j > 0, \\ U_0^{n+1} = U_N^{n+1} = 0 \end{cases}, \\ \text{for} \quad \text{n} = 0, 1, \dots, \text{M} - 1. \end{cases} \tag{3.14}$$

where

$$\mathcal{L}^* = \left[\frac{D_x^0 U_i}{\sqrt{g}}\right](\tilde{d}, t_j),$$

$$\left[\frac{D_x^0 U_i}{\sqrt{g}}\right](\tilde{d}, t_j) = \left[\frac{1}{\sqrt{g}} D_x U\right](\tilde{d}, t_j) = \frac{1-d}{(1-\tilde{d}(t_j))} D_x^+ U(\tilde{d}, t_j) - \frac{d}{\tilde{d}(t_j)} D_x^- U(\tilde{d}, t_j).$$

and the right hand side (RHS) vector f_i^{n+1} is

$$\hat{f}_i^{n+1} = \begin{cases} \left[m_i f_{i-1}^{n+1} + m_i^0 f_i^{n+1} + m_i^+ f_{i+1}^{n+1}\right], & \text{for} \quad \text{i} = 1, \dots, \text{N}/2+1, \dots, \text{N}-1, \\ 0, \quad \text{for} \quad \text{i} = \text{N}/2. \end{cases} \tag{3.15}$$

Here, the coefficients for $i = 1, \dots, N/4$ are given by

$$\begin{cases} r_i^- = \left(\dfrac{-2\varepsilon}{\hat{h}h_i} - \dfrac{a_{i-\frac{1}{2}}}{hi} + \dfrac{b_{i-\frac{1}{2}}}{2}\right) - \dfrac{1}{2\Delta t}, \\ r_i^0 = \left(\dfrac{2\varepsilon}{h_i h_{i+1}} - \dfrac{a_{i-\frac{1}{2}}}{hi} + \dfrac{b_{i-\frac{1}{2}}}{2}\right) - \dfrac{1}{2\Delta t}, \\ r_i^+ = \Delta t\left(\dfrac{-2\varepsilon}{\hat{h}h_{i+1}}\right), \\ Q_i^- = \dfrac{1}{2\Delta t}, \quad Q_i^0 = \dfrac{1}{2\Delta t}, \quad Q_i^+ = 0, \\ m_i^- = \dfrac{1}{2}, \quad m_i^0 = \dfrac{1}{2}, \quad m_i^+ = 0, \end{cases} \tag{3.16}$$

and for $i = N/4+1, \dots, N/2-1, N/2+1, \dots, 3N/4-1$,

$$\begin{cases} r_i^- = \left(\dfrac{-2\varepsilon}{\hat{h}h_i} - \dfrac{a_i}{h_i}\right), \\ r_i^0 = \left(\dfrac{2\varepsilon}{h_i h_{i+1}} + b_i\right) - \dfrac{1}{\Delta t}, \\ r_i^+ = \left(\dfrac{-2\varepsilon}{\hat{h}h_{i+1}} + \dfrac{a_i}{h_i}\right). \\ Q_i^- = 0, \quad Q_i^0 = \dfrac{1}{\Delta t}, \quad Q_i^+ = 0, \\ m_i^- = 0, \quad m_i^0 = 1, \quad m_i^+ = 0, \end{cases} \tag{3.17}$$

and for $i = 3N/4, \dots, N-1$,

$$\begin{cases} r_i^- = \left(\dfrac{-2\varepsilon}{\hat{h}_i h_i} - \dfrac{a_{dis}}{h_i} + \dfrac{b_{dis}}{2}\right), \\ r_i^0 = \left(\dfrac{2\varepsilon}{h_i h_{i+1}} - \dfrac{a_{dis}}{hi} + \dfrac{b_{dis}}{2}\right) - \dfrac{1}{\Delta t}, \\ r_i^+ = \left(\dfrac{-2\varepsilon}{\hat{h}h_{i+1}}\right), \\ Q_i^- = 0, \quad Q_i^0 = \dfrac{1}{2\Delta t}, \quad Q_i^+ = \dfrac{1}{2\Delta t}, \\ m_i^- = 0, \quad m_i^0 = \dfrac{1}{2}, \quad m_i^+ = \dfrac{1}{2}, \end{cases} \tag{3.18}$$

Finally, for $i = N/2$ the coefficients in (3.14) we have

$$\begin{cases} c_{N/2}^{-,2} = -\dfrac{1}{2h_l}, \quad c_{N/2}^{-,1} = \dfrac{2}{h_l}, \\ c_{N/2}^{0} = -\dfrac{3}{2}\left(\dfrac{1}{h_r} + \dfrac{1}{h_l}\right), \\ c_{N/2}^{+,1} = \dfrac{2}{h_l}, \quad c_{N/2}^{+,2} = -\dfrac{1}{2h_r}. \end{cases} \tag{3.19}$$

Finally, the numerical solution at the $(n+1)th$ level can be obtained by solving the tridiagonal equation. In general, we use the Shishkin mesh for the hybrid finite difference technique in the preceding sections and we will choose the special meshes by comparing piecewise-uniform S-type mesh.

E-UNIFORM CONVERGENCE OF THE FULLY DISCRETE SCHEME

We recall that, by combining the time semi-discretization scheme (3.1) obtained by applying the implicit Euler method and rearranging the terms in (3.14) obtained by applying a hybrid scheme for the spatial derivative, we have:

$$\begin{cases} U_i^0 = \phi(x_i), \quad 0 \le i \le N, \quad [\phi](\tilde{d}) \neq 0, \quad 0 < \tilde{d} < 1, \\ \begin{cases} \mathcal{L}_{hyb}^N U_i^{n+1} \equiv \left[r_i^- U_{i-1}^{n+1/2} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1}\right] + \left[Q_i^- U_{i-1}^n + Q_i^0 U_i^n + Q_i^+ U_{i+1}^{n+1}\right] \\ = \hat{f}_i^{n+1}, \quad for \quad i = 1, \dots, N/2+1, \dots, N-1, \\ = 0, \quad i = \dfrac{N}{2} \\ \mathcal{L}^* = 0, \quad x_i = \tilde{d}, \quad t_j > 0, \\ U_0^{n+1} = U_N^{n+1} = 0 \end{cases}, \\ \text{for} \quad \text{n} = 0, 1, \dots, \text{M} - 1 \end{cases} \tag{4.1}$$

where

$$\mathcal{L}^* = \left[\frac{D_x^0 U_i}{\sqrt{g}}\right](\tilde{d}, t_j),$$

$$\left[\frac{D_x^0 U_i}{\sqrt{g}}\right](\tilde{d}, t_j) = \left[\frac{1}{\sqrt{g}} D_x U\right](\tilde{d}, t_j) = \frac{1-\tilde{d}}{(1-\tilde{d}(t_j))} D_x^+ U(\tilde{d}, t_j) - \frac{d}{\tilde{d}(t_j)} D_x^- U(\tilde{d}, t_j).$$

and the coefficients are given by (3.16), (3.17), (3.18) and (3.19). An existing solver can solve the above tridiagonal system of linear algebraic equations.

Lemma *Truncation error (hybrid scheme) Let t(x) be a smooth function defined on* $\bar{\Omega}$ *also let* $r_i = r(x_i)$ *on* $\bar{\Omega}^N$. *For* $i = 1, \dots, N-1$, *the following estimates holds true:*

Theorem 4.2. *(Truncation Error) Let* $\tilde{d}(T) \le 1-\delta$ *and* $M = O(N)$. *If U is the solution of discrete problem (3.14) and u is the solution of singular problem (2.7), then we have,*

$$||\hat{U} - u||_{\bar{D}} \le CN^{-1}\ln N + C|[\phi'](\tilde{d})|M^{-1/2}. \tag{4.2}$$

Proof The proof can be obtained in the same way as in Theorem 1 of [10, 12].

Theorem 4.3. *For sufficiently large N and* $M \ge O(\ln N)$, *the solution Z of*

$$\mathcal{L}^{N,M} Z = \mathcal{L}z, \quad x_i \neq \tilde{d} \quad t_j > 0;$$

$$Z = 0, \quad (x_i, t_j) \in \partial D^{N,M}; \quad \left[\frac{D_x^0 Z}{\sqrt{g}}\right](\tilde{d}, t) = 0.$$

satisfies the following bounds

$$|Z(x_i, t_j)| \le C \frac{\prod_{n=\frac{N}{2}}^{i} \left(1 + \dfrac{h_n}{\sqrt{2T_\varepsilon}}\right)}{\prod_{n=1} \left(1 + \dfrac{h_n}{\sqrt{2T_\varepsilon}}\right)} + CN^{-1}\ln N, \quad x_i \le \tilde{d}.$$

$$|Z(x_i,t_j)| \le C \prod_{n=\frac{N}{2}}^{i} \left(1+\frac{h_n}{\sqrt{2T_\varepsilon}}\right)^{-1} + CN^{-1}\ln N, \quad x_i \ge \tilde{d}.$$

Proof It can be proved by following the steps in Theorem 3 of [13, 10].

Theorem 4.4. *Suppose* $\frac{2T}{\delta}\|a_x\| \le 1-\gamma$, $0 < \gamma < 1$, $a_x(\tilde{d}, 0) = 0$. *For sufficiently large N and* $M \ge O(\ln(N))$, *the solution Z of*

$$\mathrm{L}^{N,M} Z = \mathrm{L}z, \qquad x_i \ne d \qquad t_j > 0;$$

$$Z = 0, \quad (x_i,t_j) \in \partial D^{N,M}; \quad \left[\frac{D_x^0 Z}{\sqrt{g}}\right](\tilde{d},t) = 0,$$

satisfies the following bounds

$$|Z(x_i,t_j) - Z(x_i,t_j)| \le C\left(N^{-1}(\ln N)^2 + CM^{-1}\right) \tag{4.3}$$

Proof. It can be proved by following the steps in Theorem 4 of [10, 11].

ERROR ANALYSIS AND STABILITY

1. Stability

Lemma *(Discrete minimum principle) For the purpose of error analysis, we make the assumption that* $N \ge N_0$ *and*

$$\frac{N}{\ln N_0} \ge 2\tau_0 \alpha ||a||_\infty, \tag{5.1}$$

$$\left(||b||_\infty + \frac{1}{\Delta t}\right) \le \frac{\alpha^* N_0}{2}. \tag{5.2}$$

where $\alpha^* = \min\{\alpha_1^*, \alpha_2^*\}$, $\alpha = \max\{\alpha_1, \alpha_2\}$ *and* N_0 *is an arbitrary positive integer. Then, discrete operator* L^N_{hyb} *defined in the hybrid numerical scheme in (3.14) satisfies the discrete minimum principle, in other words, if* $\{Z_i\}$ *are the mesh functions that satisfy* $Z_0 \le 0$, $Z_N \le 0$ *and* $\mathrm{L}^N_{hyb} Z_i \ge 0$, *in* $\hat{\Omega}N$, *then* $Z_i \le 0$ *on* $\hat{\Omega}N$.

Proof. The proof can be done by following the steps in Theorem 5.3.1 of [19] Here, we remark that, if an M-matrix is associated with a hybrid finite difference scheme (3.14), then by assuming (5.1) and (5.2), we can conclude that the matrix is an M-matrix (monotone increasing). Hence, the operator in the hybrid scheme operator (3.14) satisfies the discrete minimum principle (see [22] for details).

a. *Error Analysis*

This section discusses the uniform error estimates for the numerical scheme in (3.14). We decom- pose the discrete solution (3.14) into smooth(regular), boundary and singular layer components analogous to the continuous problem (3.1) to estimate the nodal error $|\hat{U}^{n+1} - u(x_i, t_j)|$. We note that we study the error estimates separately for the outside and inside of the layer regions.

i. *Decomposition of the discrete solutions in the transformed domain*

We will estimate the nodal error $|U^{N,M}(x_i, t_n) - u(x_i, t_n)|$ by decomposing the solution $U^{N,M}(x_i, t_n)$ on the mesh $D^{N,M}$ in the following manner:

$$\underbrace{U^\pm}_{\text{Numerical solution}} = \underbrace{V^\pm}_{\text{Smooth solution}} + \underbrace{W^\pm}_{\text{Boundary component}}$$

Where the smooth (regular) component $V^\pm$ and the singular component $W^\pm$ are defined separately. Define the mesh function P_L and P_R (which is approximate p respectively to the left and right of the discontinuity $x = \tilde{d}$) to the solutions of the following discrete problems

$$\begin{cases} \mathcal{L}_{hyb}^{N,M} P_L = f(x_i, t_j), & (x_i, t_j) \in D^N \cap \bar{D}^-, \\ P_L(0, t_j) = v(0, t_j), \quad P_L(\tilde{d}, t_j) = p(\tilde{d}^-, t_j), & t_j > 0, \\ P_L(x_i, 0) = p(x_i, 0), \quad x_i \le \tilde{d}, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \mathcal{L}_{hyb}^{N,M} P_R = f(x_i, t_j), & (x_i, t_j) \in D^N \cap \bar{D}^-, \\ P_R(1, t_j) = v(1, t_j), \quad P_R(\tilde{d}, t_j) = p(\tilde{d}^+, t_j), & t_j > 0, \\ P_R(x_i, 0) = p(x_i, 0), \quad x_i \ge \tilde{d}, & . \end{cases} \tag{5.4}$$

Define the mesh function $Q_L : \bar{D}^N \cap [0, \tilde{d}] \to R$ and $Q_R : \bar{D}^N \cap [\tilde{d}, 1] \to R$ (which is approximate q respectively to the left and right of the discontinuity $x = d$) to the solutions of the following discrete problems

$$\begin{cases} \mathcal{L}_{hyb}^{N,M} Q_L = 0, \quad (x_i, t_j) \in D^N \cap \bar{D}^-, \\ \mathcal{L}^N Q_R = 0, \quad (x_i, t_j) \in \bar{D}^N \cap \bar{D}^+, \\ Q_L(0, t_j) = 0, \quad Q_L(x_i, 0) = 0, \quad x_i \le 0, \\ Q_R(x_i, 0) = 0, \quad x_i \ge \tilde{d}, \quad Q_R(1, t_j) = 0, \\ Q_R(\tilde{d}, t_j) + P_R(\tilde{d}, t_j) = Q_L(\tilde{d}, t_j) + P_L(\tilde{d}, t_j), \\ D_x^+ Q_R(\tilde{d}, t_j) + D_x^+ P_R(\tilde{d}, t_j) = D_x^- Q_L(\tilde{d}, t_j) + D_x^- P_L(\tilde{d}, t_j). \end{cases} \tag{5.5}$$

Finally, we can define the discrete solution U as

$$\begin{cases} P_L(x_i) + Q_L(x_i), \quad (x_i, t_j) \in D^N \cap \bar{D}^-, \\ P_L(\tilde{d}) + Q_L(\tilde{d}) = P_R(\tilde{d}) + Q_L(\tilde{d}), \quad x_i = \tilde{d}, \\ Q_R(x_i) + P_R(x_i), \quad (x_i, t_j) \in \bar{D}^N \cap \bar{D}^+, \end{cases} \tag{5.6}$$

In this section, we present some key lemmas to demonstrate the numerical solution's ε-uniform convergence in the discrete supremum norm. We study the error estimates in the outer layer region, inner layer region, and interior layer region separately in the following sections:

ii. The error estimate in the outer region

The error bounds associated with the smooth components are obtained in the following lemma.

Lemma 5.2. *(Error in the smooth component) Assume that $\varepsilon \le CN^{-1}$. Then under the assump- tions (5.1) and (5.2) of Lemma 5.1, the errors to the smooth layer components satisfy the following estimates*

$$\begin{cases} |V_i^L - v(x_i, t_n)| \le C\left(N^{-1} \ln N + CM^{-2} + \Delta t\right), \quad (x_i, t_n) \in \bar{D}^{N,M} \quad for \quad 1 \le i \le N/4 \cup 3N/4 \le i \le N-1, \\ |V_i^R - v(x_i, t_n)| \le C\left(N^{-2} \ln^2 N + \Delta t\right), \quad for \quad N/4 \le i \le 3N/4. \end{cases} \tag{5.7}$$

Proof: The detailed proof is given by Mukherjee and Natesan in [22, 21].

iii. The error estimate in the inner region

The error bounds associated with the singular (boundary) components are obtained in the following lemma.

Lemma 5.3. *(Error in the singular component) We assume that $\gamma \ge \alpha/2$ and $\tau_0 \ge 2/\alpha$. Then under the assumptions (5.1) and (5.2) of Lemma 5.1, the errors associated with the singular com- ponents satisfy the following estimates*

$$\begin{cases} |W_i^L - w(x_i,t_n)| \le C\left(N^{-2} + \Delta t\right), & (x_i,t_n) \in \bar{D}^{N,M} \quad for \quad N/4 \le i \le N/2-1, \\ |W_i^R - w(x_i,t_n)| \le C\left(N^{-2} + \Delta t\right), & for \quad 3N/4 \le i \le N-1. \end{cases} \tag{5.8}$$

where, $W_i^L = W_{1,i}^L + W_{2,i}^L$ and $W_i^R = W_{1,i}^R + W_{2,i}^R$.

Proof: The detailed proof is given by Mukherjee and Natesan in [22, 21]. Furthermore, in the following lemma, we obtain the error bounds for the non-smooth components.

Lemma 5.4. *Suppose* $\gamma = \alpha/2$ *Then, under the assumptions (5.1) and (5.2) of Lemma* (3.9)*, the errors associated with the hybrid scheme (3.14)*

$$|U_i - u(x_i, t_n)| \le C\ \ N^{-2} + \Delta t\ \ , \qquad for \qquad 1 \le i \le N/4 \cup 3N/4 \le i \le N-1. \tag{5.9}$$

Proof: The detailed proof is given by Majumdar and Natesan in [19].

iv. The error estimate in the interior layer region

On the domain $\bar{\Omega}^N = \{x_i\}^{0\ N}$, we introduce the following two mesh functions

$$S_i = \prod_{k=1}^{i} \left(1 + \frac{\gamma h_k}{\varepsilon}\right), \quad \text{for} \quad 1 \le i \le N/2$$

$$\Lambda_i = \prod_{j=1}^{i} \left(1 + \frac{\gamma h_k}{\varepsilon}\right), \quad \text{for} \quad N/2+1 \le i \le N-1$$

(with the standard convention of $S_0 = 1$ and $\Lambda_N = 1$) where γ is a positive constant.

Lemma 5.5. *(Error in the interior component)*

$$|Z - z(x_i,t_n)| \le \begin{cases} C\left(N^{-2} + \Delta t\right), & for \quad 1 \le i \le N/4 \cup 3N/4 \le i \le N-1, \\ C\left(N^{-2}\ln^2 N + \Delta t\right), & for \quad N/4 \le i \le 3N/4. \end{cases} \tag{5.10}$$

And

$$\begin{cases} |Z_L - z(x_i,t_n)| \le CN^{-1}\ln N + CM^{-1} + \Delta t, & (x_i,t_n) \in \bar{D}^{N,M} \quad for \quad N/2, \\ |Z_R - z(x_i,t_n)| \le C\left(N^{-2}\ln^2 N + \Delta t\right), & for \quad N/2. \end{cases} \tag{5.11}$$

Proof: The detailed proof is given by Mukherjee and Natesan in [18, 23].

The above technical lemmas are used in the following lemma to obtain the bound for $|U_i - u(x_i, t_n)|$ in the interior layer regions.

Lemma 5.6. *Suppose* $\gamma = \alpha/2$ *Then, under the assumptions (5.1) and (5.2) of Lemma* (3.9)*, the errors associated with the hybrid scheme (3.14)*

$$|U_i - u(x_i, t_n)| \le C\ \ N^{-2}\ln^2 N + \Delta t\ \ , for\ N/4 + 1 \le i \le 3N/4 - 1. \tag{5.12}$$

Proof: The detailed proof is given by Majumdar and Natesan in [19, 22, 21].

Theorem 5.7. *(The main convergence result)*

Suppose that $N \ge N_0$ *satisfies the condition given in (5.1), (5.2) and* $\varepsilon \le CN^{-1}$*. Then, if* $\gamma \le \alpha/2$ *and* $\tau_0 \ge 2/\gamma$*, the respective solution u and U of* (2.1) *and hybrid numerical scheme (3.14) satisfy the following error bounds at time level* t_n*;*

$$|\hat{U} - u(x_i,t_n)| \le \begin{cases} C\left(N^{-2}\ln^2 N + \Delta t\right), & for \quad 1 \le i \le N/4 \cup 3N/4 \le i \le N-1, \\ C\left(N^{-2} + \Delta t\right), & for \quad N/4 \le i \le 3N/4 \cup 3N/4 \le i \le N-1. \end{cases} \tag{5.13}$$

Proof: We split the proof into three different cases depending on the mesh point $x_i \in \hat{\Omega}_x^N$:

Case 1 (Outer region): For $1 \leq i \leq N/4$ and $N/2 \leq 3N/4 - 1$. In this case, we consider the mesh points $\{x_i\}$, for $N/4 \leq i \leq N/2 - 1$ and $3N/4 \leq i \leq N - 1$. By using Lemma 5.2 and Lemma 5.3 in

$$|\left(\hat{U} - u\right)(x_i, t_n)| \leq \begin{cases} |(V_L - v_L)(x_i, t_n)| + |(W_L - w_L)(x_i, t_n)|, & for \quad 1 \leq i \leq N/4 \cup 3N/4 \leq i \leq N-1, \\ |(V_R - v_R)(x_i, t_n)| + |(W_R - w_R)(x_i, t_n)|, & for \quad N/4 \leq i \leq 3N/4. \end{cases} \tag{5.14}$$

we get the required error bounds in the outer region.

Case 2 (Inner region): Here, we consider the mesh points $\{x_i\}$, for $1 \leq i \leq N/4 - 1$ and $N/2 \leq i \leq 3N/4 - 1$. Now, for $1 \leq i \leq N/4 - 1$, by applying Taylor's series expansion with the integral form of remainder term and the bounds of the derivatives given in Lemma 5.2 and Lemma 5.3, we get

$$\begin{aligned} |L_\varepsilon^{*N}\left(|\hat{U} - u(x_i)|\right)| &\leq Ch \int_{x_{i-1}}^{x_{i+1}} \left[\varepsilon|u^4(s)| + |u^3(s)|ds\right] \\ &\leq Ch^2 + ch\left(\varepsilon^{-2} + \varepsilon^{-1}\right) \leq \left(\exp(-\frac{\alpha x_{i-1}}{\varepsilon})\exp(-\frac{\alpha x_i}{\varepsilon})\right) \\ &\leq \left[h^2 + \frac{h^2}{\varepsilon^3}\exp(-\frac{\alpha x_i}{\varepsilon})\right]. \end{aligned}$$

Similarly, for $N/2 + 1 \leq 3N/4 - 1$ we have

$$|L_\varepsilon^{*N}\left(|\hat{U} - u(x_i)|\right)| \leq \left[h^2 + \frac{h^2}{\varepsilon^3}\exp(-\frac{\alpha x_{i-\tilde{d}}}{\varepsilon})\right].$$

Case 3 (Interior layer region):

Here, we need to find the bound for the error estimate $|Z - z(x_i, t_n)|$ for $i = N/2$. In general, the estimate for $\hat{U} - u(x_i, t_n)$ follows from Lemma 5.2, Lemma 5.3 and Lemma 5.5 with applying the triangle inequality to the problem

$$\hat{U} - u \quad (x_i, t_n) = (V - v)(x_i, t_n) + (W - w)(x_i, t_n) + (Z - z)(x_i, t_n)$$

NUMERICAL EXAMPLES, RESULTS AND DISCUSSION

In this section, we illustrate the numerical experiments for the theoretical findings presented in the previous sections. The uniform two-mesh (double-mesh) global differences $E^{N,\Delta t}$ and uniform orders of global convergence $P_\varepsilon^{N,\Delta t}$ are calculated. Then for each ε, we can calculate the maximum point-wise error by

$$E_\varepsilon^{N,\Delta t} := \max_{(x_i,t_n)\in \bar{D}_\varepsilon^{N,M}} |U_\varepsilon^{N,\Delta t}(x_i, t_n) - U_\varepsilon^{2N,\frac{\Delta t}{2}}(x_i, t_n)|$$

and the corresponding order of convergence by,

$$P^{N,\Delta t} = log_2\left(\frac{E_\varepsilon^{N,\Delta t}}{E_\varepsilon^{2N,\frac{\Delta t}{2}}}\right).$$

Now, for each N and Δt, define $E^{N,\Delta t} = \max_\varepsilon E_\varepsilon^{N,\Delta t}$ as the ε-uniform maximum point-wise error and the corresponding local ε-uniform order of convergence is defined by

$$P^{N,\Delta t} = log_2\left(\frac{E^{N,\Delta t}}{E^{2N,\frac{\Delta t}{2}}}\right).$$

The maximum point-wise errors $E_\varepsilon^{N,\Delta t}$, the corresponding order of convergence $P_\varepsilon^{N,\Delta t}$ and the ε-uniform errors $E^{N,\Delta t}$, the corresponding ε-uniform order of convergence $P^{N,\Delta t}$, for Example 6.2 and Example **??** are presented in different tables for various values of ε and N. Consider the following singularly perturbed parabolic problem with discontinuous initial conditions:

Example 6.1. *Consider the following singularly perturbed parabolic convection-diffusion problems with discontinuous initial conditions:*

$$\begin{cases} u_t - \varepsilon u_{xx} + a(x,t)u_x = 4x(1-x)t + t^2, & (x,t) \in (0,1) \times (0, \frac{1}{2}], \\ u(x,0) = -2; \quad 0 \le x < 0.2, \quad u(x,0) = 2; & 0.2 \le x \le 1, \\ u(0,t) = -2, \quad u(1,t) = 1, & 0 < t \le 0.5. \end{cases} \tag{6.1}$$

where

$$a(x,t) = \frac{(0.9)^2 - (x-0.2)^2}{4}$$

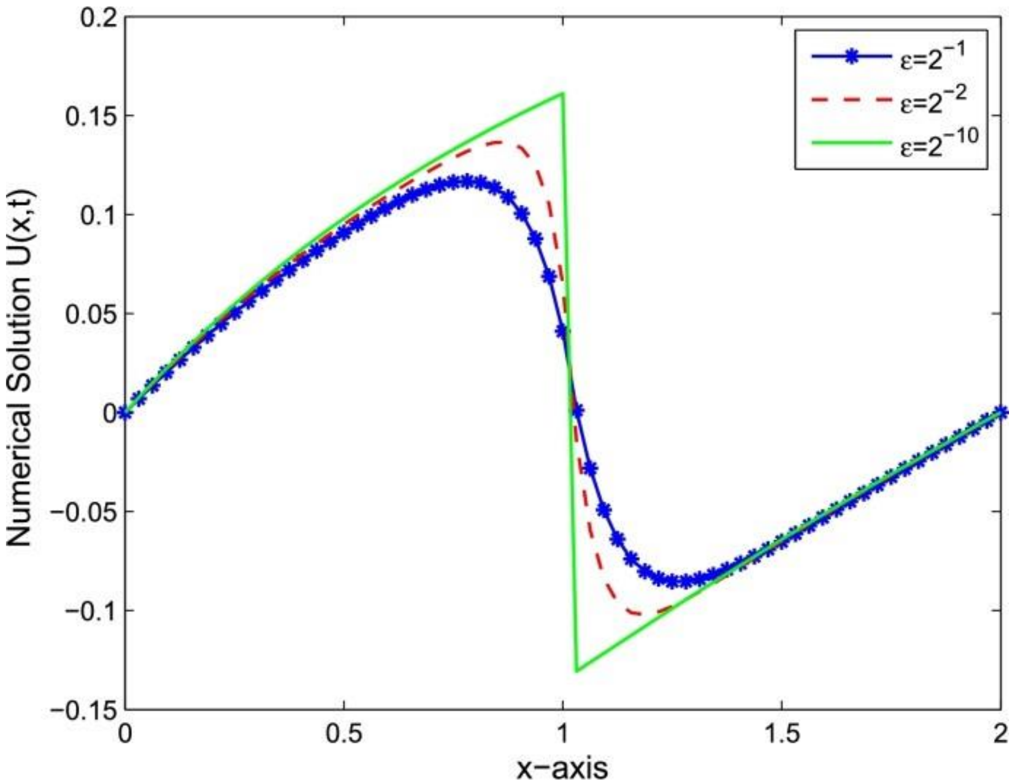


Figure 1: Exact solution for Example 1 on the mesh for $N = 256$ of Example 6.1

Note that $a_x(\tilde{d}, 0) = 0$.

$$\tilde{d}(t) = \frac{1.1 - 0.7\exp(-9t/20)}{1 + \exp(-9t/20)}$$

Hence, we first examine if this transformation is needed if $a = a(x, t)$. In table 2, we see that, without the mapping, the method is not-parameter-uniform. Therefore, example 1 is now approx- imated with the numerical scheme (discrete problem) (3.10) and (3.11) proposed in this work. In addition, the computed approximations to $\tilde{y}$ and u are displayed in figures (5) and (6). Again, the maximum double-mesh global differences are given in table 2. These numerical results are in agreement with the theorem 4.3.

Example 6.2. *Consider the following singularly perturbed parabolic problem with discontinuous initial conditions:*

$$\begin{cases} u_t + \varepsilon u_{xx} + a(x)u_x(x,t) = f(x,t), & (x,t) \in (0,1) \times (0,1], \\ u(0,t) = t^2, \quad u(1,t) = 0, & 0 \le t \le 1, \end{cases} \tag{6.2}$$

The initial condition is

$$u(x,0) = \begin{cases} 0.5x, & 0 \le x \le \tilde{d}, \\ x - 1, & \tilde{d} \le x \le 1 \end{cases} \tag{6.3}$$

where the source term

$$f(x,t) = \begin{cases} 3t^2(1+x^2)/2, & 0 \le x \le 1/2 \text{ and } 0 \le t \le 1, \\ t^2(1+x^2), & 1/2 < x \le 1 \text{ and } 0 \le t \le 1. \end{cases} \tag{6.4}$$

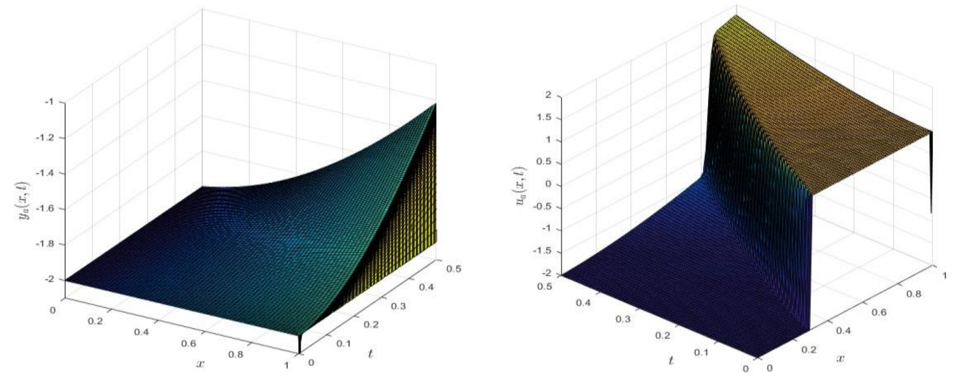


Figure 2: Surface plot of numerical approximation to y and u with $\varepsilon = 2^{-12}$ and $N = M = 64$ for Example 6.1.

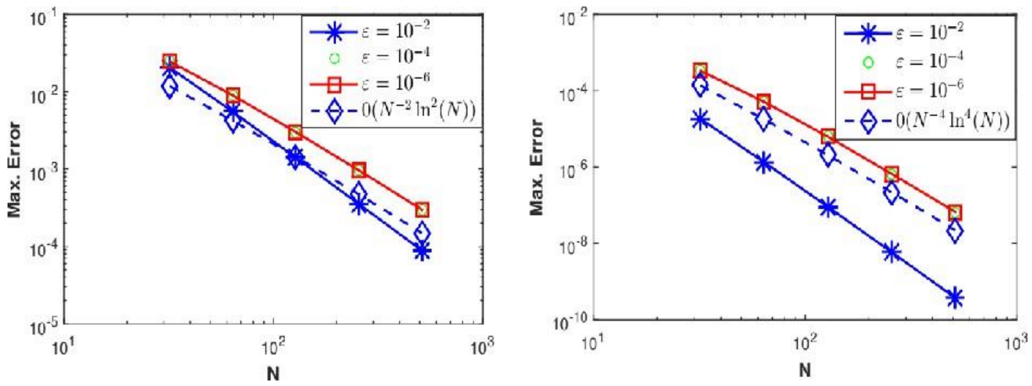
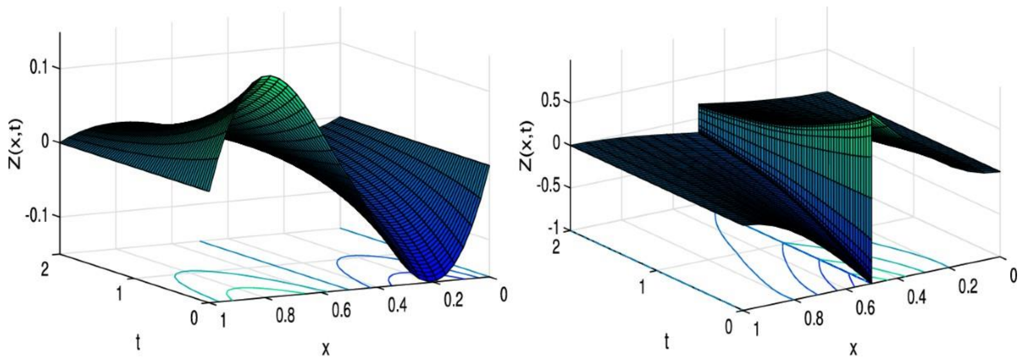


Figure 3: Loglog plot of ε-uniform maximum point-wise errors for Example 6.1 and 6.2.



(a) $\varepsilon = 10^{-1}$ (b) $\varepsilon = 10^{-5}$

Figure 4: Loglog plot of ε-uniform maximum point-wise errors for Example 6.2.

and the convective term coefficients are

$$a(x) = \begin{cases} 1 + x(1-x), & 0 \le x \le 0.5, \\ 2 + x(1-x), & 0.5 < x \le 1. \end{cases} \tag{6.5}$$

We briefly state the output of our numerical results with this example of discontinuous initial data. For this test example, plots of $u^{N,M}$ and $y^{\sim N,M} = u^{N,M} + \tilde{S}$ are given for the sample of values, and the set of a parameter as

$$\varepsilon = \{10^0, 10^{-2}, 10^0, \ldots, 10^{-26}\}$$

$N = M = 64$

For this test example, plots of $u^{N,M}$ and $y^{\sim N,M} = u^{N,M} + \tilde{S}$ are given for the sample of values, and the set of a parameter as $\varepsilon = \{10^0, 10^{-2}, 10^0, \ldots, 10^{-26}\}$ and $N = M = 64$.

CONCLUSIONS

This paper studies a hybrid technique for solving 1D singularly perturbed parabolic convection- diffusion problems with discontinuous initial conditions by considering a proper combination of central differences and a midpoint upwind technique. The stability and convergence of the proposed approach have been investigated and ε-uniform error estimates have been obtained. The method is second-order convergent. Two examples are studied to verify the order of convergence and accuracy of the theoretical error estimates.

From the numerical experiments, we made the following observations about the newly proposed scheme for solving a class of singularly perturbed discontinuous initial conditions of the form (2.1). Firstly, it is observed that the ε-uniform errors (i.e.,ε^N or E^N) obtained in Tables 1 and 2, de- crease monotonically as N increases. This ensures that the hybrid numerical scheme is ε-uniformly convergent.

Secondly, from the numerical results displayed in Tables 1 and 2, it is clear that the ε-uniform order of convergence of the hybrid scheme is nearly one as N increases, where as the proposed hybrid scheme converges ε-uniformly with almost second-order accuracy. These observations are in excellent agreement with the theoretical results obtained for the proposed hybrid scheme. More precisely, we notice that the current hybrid scheme yields higher-order accurate numerical results, particularly for larger values of ε.

Table 1: *Maximum point-wise errors and order of convergence for Example 6.1.*

$\varepsilon \downarrow$	Number of mesh points $N = M$				
	N=M=16	N=M=32	N=M=64	N=M=128	N=M=256
10^0	4.002e-4	1.001e-4	2.500e-5	6.250e-6	1.562e-6
	2.0001	2.0006	2.0002	2.0003	2.0001
10^{-2}	7.968e-4	2.012e-4	5.032e-5	1.258e-5	3.146e-6
	1.9858	1.9991	1.9998	2.0000	2.0000
10^{-4}	1.611e-3	4.072e-4	1.023e-4	2.560e-5	6.401e-6
	1.9843	1.9929	1.9987	1.9996	1.9999
10^{-6}	3.020e-3	7.848e-4	1.982e-4	4.966e-5	1.243e-5
	1.9441	1.9857	1.9964	1.9982	1.9998
10^{-8}	5.205e-3	1.495e-3	3.880e-4	9.793e-5	2.454e-5
	1.8000	1.9459	1.9862	1.9965	1.9991
10^{-10}	3.374e-3	1.902e-3	7.464e-4	1.936e-4	4.886e-5
	0.8266	1.3499	1.9468	1.9864	1.9966
10^{-12}	1.691e-3	1.008e-3	5.089e-4	2.064e-4	7.276e-5
	0.7462	0.9860	1.3024	1.5039	1.4941
10^{-14}	8.466e-4	5.111e-4	2.568e-4	1.039e-4	3.660e-5
	0.7280	0.9929	1.3054	1.5054	1.4923
10^{-16}	4.236e-4	2.574e-4	1.290e-4	5.215e-5	1.836e-5

	0.7189	0.9964	1.3070	1.5062	1.4914
10^{-18}	2.119e-4	1.292e-4	6.466e-5	2.612e-5	9.193e-6
	0.7143	0.9981	1.3077	1.5066	1.4910
10^{-20}	1.060e-4	6.469e-5	3.307e-5	1.572e-5	6.309e-6
	0.7120	0.9683	1.0730	1.3169	1.9464
.	.	.	.	.	.
10^{-24}	2.650e-5	1.631e-5	9.064e-6	4.682e-6	2.294e-6
	0.7006	0.8470	0.9530	1.0294	1.1270
.	.	.	.	.	.
10^{-32}	1.656e-6	1.035e-6	5.823e-7	3.076e-7	1.576e-7
	0.6776	0.8303	0.9208	0.9644	0.9884
$E^{N,M}$	3.462E-02	3.546E-02	1.531E-02	4.269E-03	1.676e-7
	1.440	1.570	1.567	1.322	1.041
$P^{N,M}$	1.656e-6	1.035e-6	5.823e-7	3.076e-7	1.576e-7
	1.696	1.655	1.675	1.5644	1.6884

Table 2: *Maximum point-wise errors and order of convergence for Example 6.2.*

$\varepsilon \downarrow$	Number of mesh points $N = M$				
	N=M=16	N=M=32	N=M=64	N=M=128	N=M=256
10^{-1}	0.9875	0.9878	0.9947	0.9972	0.989
10^{-2}	0.7629	0.7929	0.8127	0.8244	0.9428
10^{-3}	1.6417	1.6903	1.7062	1.7178	1.8456
	0.7554	0.7901	0.8090	0.8254	0.8143
10^{-5}	1.6419	1.6903	1.7071	1.7177	1.9428
.	0.7540	0.7907	0.8088	0.8254	0.9306
10^{-9}	1.6414	1.6904	1.7073	1.7178	1.9856
10^{-1} 0	.	.	.	.	.
	0.7538	0.7906	0.8088	0.8254	0.8452
$E^{N,}$ M	1.6413	1.6904	1.7073	1.7178	1.7856
	0.7538	0.7906	0.8088	0.8254	0.9856
P N,M	1.6413	1.6904	1.7073	1.7178	1.7156
	1.6481e-2	**1.0402e-2**	**5.8462e-3**	**4.3853e-3**	**1.9118e-3**
$E^{N,}$ M	**0.7829**	**0.7939**	**0.8327**	**0.8344**	**0.8244**
extp	**1.1578e-3**	**3.4876e-4**	**1.1596e-4**	**3.4143e-5**	**9.9732e-6**
P N,M	**1.6523**	**1.6904**	**1.7182**	**1.7378**	**1.8258**
extp					

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