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FUZZY TOPOLOGICAL MODULES INDUCED BY FUZZY PSEUDO NORMED MODULE

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Abstract
In this article, we give defines about fuzzy pseudo norm module space and fuzzy metric module space which induces by this space. Complete metric space was proved. Also we are defining fuzzy topological module space which induces by fuzzy metric module space.

Keyword: *Fuzzy pseudo norm module space, fuzzy metric module space, fuzzy scalar, complete fuzzy metric module space, fuzzy topological module space*

INTRODUCTION:

In 1965 [10], L. Zadeh gave the notion of fuzziness. In 1968, C. Chang [1] gave the definition of fuzzy topology. Xia and Guo (2004) [9] using fuzzy scalars to measure the distances between fuzzy points, which is consistent with the theory of fuzzy linear spaces. Deb Ray, A [5, 6] introducing the concept of fuzzy topological ring, fuzzy continuous function and studied left fuzzy topological ring.

Basim Mohammed Melgat and Munir AL-Khafaji [2, 3 and 4] (2019) gave some results of fuzzy top. ring. In 2020 [7] Mohammed M. Ali, introduce fuzzy topological module space and fuzzy topological submodule space

In the present work we defining fuzzy pseudo norm module space and defining fuzzy metric module space which induces by fuzzy norm module space and using the concept of fuzzy scalar to measure the distances between fuzzy points. Also we are giving fuzzy topological module space which induces by fuzzy metric module space.

To rich the article some fundamental of fuzzy set and point are gave below. The symbol J will denote the closed interval $[0,1]$. Let M be a non-empty set.

Definition [10] 1.1

The term fuzzy set in a set M is a map $A: M \to J$, that is, an element of J^M . let $E \in J^M$. $m \in M$, we symboled by $E(m)$ or m_α of the membership degree of m in E . If $E(m)$ belong to $\{0, 1\}$, then E is a crisp set in M.

Definition [8]**1.2**

A pair (M , τ_M) , where M is R- module and τ_M a topology on , is called a topological module if the following maps are continuous:

1) $R \times R \to R , (r , k) \to r + k.$
2) $R \to R , r \to -r$
3) $R \times M \to M , (m , k) \to m. k$

Definition [1] 1.3

A class $\tau_F \in J^M$ of fuzzy set is called a fuzzy topology for M if the following are hold:

1) $\emptyset , M \in \tau_F$
2) $\forall A , B \in \tau_F \to A \wedge B \in \tau_F$
3) $\forall (A_j)_{j\in J} \in \mu \to \vee_{j\in J} A_j \in \tau_F$

(M, τ_F) is called fuzzy topological space. The set E is fuzzy open if $E \in \tau_F$. and complement of E is an fuzzy closed.

Definition1.4 [7]

Let R be a ring and let M be a left R-module. A fuzzy set E in M is called a fuzzy left R-module if for each $m, n \in M$ and $r \in R$:

(1) $E(m + n) \geq min \{E(m), E(n)\}$.
(2) $E(m) = E(m^{-1})$.
(3) $E(rm) \geq E(m)$.
(4) $E(0) = 1$.

Definition 1.5[[7]

Let R be a fuzzy topological ring ,the set M is said to be left fuzzy topological module on the fuzzy topological ring R if:

(1) E left fuzzy module on R.
(2) E is a fuzzy topology compatible with the stricture of fuzzy group on E and

satisfies the following axiom :

The mapping $R \times M \longrightarrow M$ defined by $(\lambda, m) \longrightarrow \lambda m$ is a fuzzy continuous

Definition 1.6 [9]

Let $P_F(M)$ be the set of all the fuzzy point m_α of M. Especially, if $M = \mathbb{R}$ (the real space), we say that fuzzy scalars instead of fuzzy points. We symboled by $S_F(\mathbb{R})$ for all fuzzy scalars set and we symboled by $S_F^+(\mathbb{R})$ for all positive fuzzy scalars set.

Definition 1.7 [9]

Suppose that M is a fuzzy linear space. (M, π_F) is said to be a fuzzy normed linear space if the following mapping $\pi_F: P_F(M) \to S_F^+(\mathbb{R})$ satisfies:

1) $\pi_F(m_\alpha) = 0$ iff $m = 0$ and $\alpha = 1$
2) For any $r \in \mathbb{R}$ and$m_\alpha \in M$ then $\pi_F(rm_\alpha) = |r|\pi_F(m_\alpha)$
3) For any $m_\alpha, n_\beta \in M,$ $\pi_F(m_\alpha, n_\beta) \leq \pi_F(m_\alpha) + \pi_F(n_\beta)$

Definition 1.8 [9]
Suppose that (M,π_F) is a fuzzy linear space defined on M. The fuzzy norm π_F of any fuzzy point m_α in M is defined by:

$$\pi_F(m_\alpha) = (\pi(m), \alpha), \qquad \forall\, m_\alpha \in M$$

Where $\pi(m)$ is the norm of m defined in (M,π).

Definition 1.9 [9]
Suppose M is a nonempty set and $d_F: P_F(M) \times P_F(M) \to S_F^+(\mathbb{R})$ is a mapping. $(P_F(M), d_F)$is said to be a fuzzy metric space if for any $\{m_\alpha, n_\beta,\ h_\gamma\} \subset P_F(R)$, d_F satisfies the following three conditions,

1) $d_F(m_\alpha, n_\beta) \geq 0$ and $d_F(m_\alpha, n_\beta) = 0$ iff $m = n$ and $\alpha = \beta = 1$
2) $\mathrm{d_F}(m_\alpha, n_\beta) = \mathrm{d_F}(\mathrm{n}_{\beta,}, \mathrm{m}_\alpha)$
3) $d_F(m_\alpha, n_\beta) \leq d_F(m_\alpha, h_\gamma) + d_F(h_{\gamma,} n_\beta)$

Definition 1.10 [9]
Suppose (M,d) is an ordinary metric space. The distance of any two fuzzy points m_α, n_β in $d_F(R)$ is defined by

$$d_F(m_\alpha, n_\beta) = (d(m,n), \min\{\alpha, \beta\})$$

Where $d(m,n)$ is the distance between m and n defined in (M,d).

Fuzzy Normed Module Space

Definition 2.1
Suppose that M be an R- module. (M, π_{FM}) is said to be a fuzzy pseudo normed R-module space if the following mapping $\pi_{FR}: P_F(M) \to S_F^+(\mathbb{R})$ satisfies:

1) $\pi_{FM}(m_\alpha) = 0$ iff $r = 0$ and $\alpha = 1$
2) For any $m_\alpha, k_\beta \in M,\ \pi_{FM}(m_\alpha - k_\beta) \leq \pi_{FM}(m_\alpha) + \pi_{FM}(k_\beta)$
3) For any $m_\alpha, k_\beta \in M,\ \pi_{FM}(m_\alpha . k_\beta) \leq \pi_{FM}(m_\alpha).\pi_{FM}(k_\beta)$
4) For each $r_\beta \in R$ and $m_\alpha \in M$ then $\pi_{Fm}(r_\beta m_\alpha) = |r_\beta| \pi_{FM}(m_\alpha)$

Definition 2.2
Suppose that (M, π_M) is a pseudo normed R-module space defined on M then (M, π_{FR}) is a fuzzy normed module space with π_{FM} is a mapping from M to $S_F^+(\mathbb{R})$ defined by:

$$\pi_{FM}(m_\alpha) = (\pi_M(m), \alpha), \qquad \forall\, m_\alpha \in M$$

Where $\pi_R(m)$ is the norm of m defined in (M, π_M).

Example 2.3
Let$(\mathbb{Z}, +, .)$be a commutative ring and (R^n, π_R) be a linear normed $\mathbb{Z}$ -module space defined in R^n (n-dim. Euclidean space). The fuzzy norm of $m_\alpha \in R^n$, is defined by $\pi_{FM}(m_\alpha) = h_\alpha$ where $h_\alpha \in S_F^+(\mathbb{R})$ such that $h = \pi_M(r)$. Then $(P_F(R^n), \pi_{FM}(m_\alpha))$ is a fuzzy normed module space

Theorem 2.4
Suppose $(P_F(M), \pi_{FM}(m))$ be a fuzzy normed R-module space, then the function $d_{FR}(m_\alpha, k_\beta) = \pi_{FR}(m_\alpha - k_\beta) = (\pi_M(m-k), \min\{\alpha, \beta\})$ satisfies the fuzzy metric axioms, thus $(P_{FM}(M), d_{FM})$ is a fuzzy metric R-module space.

Proof

1) $$d_{FM}(m_\alpha, k_\beta) = 0 \Leftrightarrow \pi_{FM}(m_\alpha - k_\beta) = 0$$
$$\Leftrightarrow\ m_\alpha = k_\beta\ \Leftrightarrow m = k \text{ and } \alpha = \beta = 1$$

Thus, $d_{FM}(m_\alpha, k_\beta) = 0$ iff $m = k$ and $\alpha = \beta = 1$

2) $$d_{FM}(m_\alpha, k_\beta) = \pi_{FM}(m_\alpha - k_\beta) = \pi_{FM}(k_\beta - m_\alpha) = d_{FM}(k_\beta, m_\alpha)$$
3) $$d_{FM}(m_\alpha . k_\beta) = \pi_{FM}(m_\alpha . k_\beta) \leq \pi_{FM}(m_\alpha).\pi_{FM} k_\beta)$$
$$\leq d_{FM}(m_\alpha).d_{FM} k_\beta)$$
4) $$d_{FM}(m_\alpha, k_\beta) = \pi_{FM}(m_\alpha - k_\beta) = \pi_{FM}(m_\alpha - h_\gamma + h_\gamma - k_\beta)$$
$$= \pi_{FM}\left((m_\alpha - h_\gamma) - (k_\beta - h_\gamma)\right)$$
$$\leq \pi_{FM}(m_\alpha - h_\gamma) + \pi_{FM}(k_\beta - h_\gamma)$$
$$= d_{FM}(m_\alpha, h_\gamma) + d_{FM}(h_\gamma, k_\beta)$$

Example 2.5
Let$(\mathbb{Z}, +, .)$be a commutative ring and $(R^n, +, ., d_M)$ be a linear normed $\mathbb{Z}$ -module space defined in R^nLet $(R^n, +, ., d_R)$ be a normed module space defined in R^n (n-dim. Euclidean space). The distance between two fuzzy point $m_\alpha, k_\beta \in$

$P_F(R^n)$, is defined by $d_{FM}(m_\alpha, k_\beta) = h_\rho$, where $h_\rho \in S_F^+(\mathbb{R})\ s.t\ h = d_R(m,k)\ and\ \rho = \min\{\alpha,\beta\}$. Then $(P_F(R^n), d_{Fm})$ is a fuzzy metric R-module space

Definition 2.6

Suppose $(P_F(M), d_{FM})$ is the fuzzy metric R-module space. We define a fuzzy open ball $B_F(m_\alpha, r_\rho)$ where m_α is the center of B_F with membership α and radius $r_\rho \in S_F^+(\mathbb{R})$ as:

$$B_F(m_\alpha, r_\rho) = \{k_\beta \in P_F(R): d_{FR}(m_\alpha, k_\beta) \leq r_\rho\}$$

Definition 2.7

Suppose $(P_F(M), d_{FM})$ is the fuzzy metric R-module space. A fuzzy set E is said to be fuzzy open set if for each m_α s.t $\alpha \leq E(m)$ there exist a fuzzy open ball $B_F(m_\alpha, r_\rho), \alpha \leq B(m), r_\rho \in S_F^+(\mathbb{R})$ s.t $B_F(m_\alpha, r_\rho) \leq E(m)$

Theorem 2.8

Suppose $(P_F(R), d_{FR})$ is the fuzzy metric R-module space. then $(P_F(M), \tau_F)$ is the fuzzy topological R-module space induced by $(P_F(M), d_{FM})$ where τ_F is defined by

$$\tau_F = \{E \leq P_F(M): E \text{ is fuzzy open set in } (P_F(M), d_{FM})\}$$

Proof

1) Let $r_0 \in \emptyset$ and $k_\alpha \in S_F^+(\mathbb{R})$ define $B_F = \{h_0: d(r_0, h_0) < k_\alpha\}$ is a fuzzy open ball, implies $r_0 \in B_F \subseteq \emptyset$. Thus $\emptyset \in \mu$
2) Let $A, C \in \tau_F$ and $r_\alpha \in A \cap C$, then $\alpha = \min\{A(r), C(r)\}$ implies $\alpha \leq A(r)$ and $\alpha \leq C(r)$. Put $B(r) = \min\{A(r), C(r)\}$ then there exists fuzzy open ball $B_F(r_\alpha, k_\beta)$ where $k_\beta \in S_F^+(\mathbb{R})$ implies $r_\alpha \leq B_F(r_\alpha, k_\beta) \leq A \cap C$. thus $A \cap C \in \tau_F$
3) Let $A_j \in \mu, j \in J$ and $r_\alpha \in \bigcup_{j\in J} A_j$, then $\alpha \leq \sup\{A_j(r), j \in J\}$ implies that there exists $j \in J$ s.t $\alpha \leq A_j(r)$. Since $A_j \in \tau_F$, then there exists fuzzy open ball $B_F(r_\alpha, k_\beta)$ where $k_\beta \in S_F^+(\mathbb{R})$ s.t $\alpha \leq B(r) \leq A(r)$ then $B_F(r_\alpha, k_\beta) \leq \sup\{A_j(r), j \in J\}$, implies $\alpha \leq B(r) \leq \sup\{A_j(r), j \in J\}$. Thus $\bigcup_{j\in J} A_j \in \tau_F$

Let's verify that the fuzzy R-module operations are fuzzy continuous in fuzzy topological τ_{FM} defined by this fuzzy metric d_{FM}. Let $a_\theta, b_\mu \in M$ and $\epsilon_\rho \in S_F^+(\mathbb{R})$, put $\delta_\rho = min\{\frac{\epsilon_\rho}{2}, \frac{\epsilon_\rho}{|r_\gamma|}\}$.

Let $m_\alpha, n_\beta \in M$, s.t $d_{FR}(m_\alpha - a_\theta) \leq \delta_\rho$ and $d_{FM}(n_\beta - b_\mu) \leq \delta_\rho$.

$$d_{FM}(m_\alpha - n_\beta, a_\theta - b_\mu) = \pi_{FM}((m_\alpha - n_\beta) - (a_\theta - b_\mu))$$
$$= \pi_{FM}((m_\alpha - a_\theta) - (n_\beta - b_\mu)) \leq \pi_{FM}(m_\alpha - a_\theta) + \pi_{FM}(n_\beta - b_\mu)$$
$$= d_{FM}(m_\alpha - a_\theta) + d_{FM}(n_\beta - b_\mu) \leq \delta_\rho + \delta_\rho \leq \epsilon_\rho$$

Let $m_\alpha, n_\beta \in M$, s.t $d_{FR}(m_\alpha - a_\theta) \leq \delta_\rho$ and $d_{FM}(n_\beta - b_\mu) \leq \delta_\rho$and hence, $\pi_{FM}(m_\alpha) - \pi_{FM}(a_\theta) \leq \pi_{FM}(m_\alpha - a_\theta) \leq \delta_\rho$ i.e $\pi_{FM}(m_\alpha) = \pi_{FM}(a_\theta) + \delta_\rho$

$$d_{FM}(m_\alpha.n_\beta, a_\theta.b_\mu) = \pi_{FM}((m_\alpha.n_\beta) - (a_\theta.b_\mu))$$
$$= \pi_{FM}((m_\alpha.n_\beta) - (m_\alpha b_\mu) + (m_\alpha b_\mu) - (a_\theta.b_\mu))$$
$$\leq \pi_{FM}(m_\alpha(n_\beta - b_\mu) + b_\mu(m_\alpha - a_\theta))$$
$$\leq \pi_{FM}(m_\alpha)\pi_{FM}(n_\beta - b_\mu) + \pi_{FM}(b_\mu)\pi_{FM}(m_\alpha - a_\theta)$$
$$\leq (\pi_{FM}(a_\theta) + \delta_\rho)\delta_\rho + \delta_\rho \pi_{FM}(b_\mu)$$
$$\leq (2\pi_{FM}(a_\theta) + \pi_{FM}(b_\mu))\delta_\rho \leq \epsilon_\rho$$

Let $m_\alpha, n_\beta \in M$ and $r_\gamma \in R$

$$d_{FM}(r_\gamma(m_\alpha, n_\beta)) = \pi_{FM}(r_\gamma(m_\alpha, n_\beta)) = |r_\gamma|\pi_{FM}(m_\alpha, n_\beta)$$
$$= |r_\gamma| d_{FM}(m_\alpha, n_\beta)$$
$$\leq \epsilon_\rho$$

Hence, the operations are continuous in the fuzzy R-module M with fuzzy topology specified by fuzzy metric. Thus, any fuzzy pseudo normed module is fuzzy topological module.

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