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THE DERIVED EXPLICIT FOURTH-STAGE SECOND-ORDER RUNGE-KUTTA METHOD IS CONSISTENT AND CONVERGENT.

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Abstract
The purpose of this paper is to analyze the consistency and convergence of an explicit fourth-stage second-order Runge-Kutta method derived using Taylor series expansion by varying parameters. The analysis revealed that the method is consistent and convergent. These properties are key and vital for any numerical method to possess for it to be capable of handling initial value problems. The implementation of this method on initial-value problems has been done in previous paper, and it revealed that the method compared favorably well with the other existing explicit Runge Kutta method of the same order. Hence, as a result of the convergence and consistency of the method, it will definitely be reliable and dependable.

Keywords: *Consistency, Convergence, Explicit, Runge-Kutta Methods, Linear and non- linear equations, Taylor series, Parameters, Initial-value Problems.*

1.0 INTRODUCTION

Some of the Runge Kutta methods derived today do not possess the properties of convergence and consistency, hence, they are not capable enough to handle problems the way they ought to. This paper successfully analyzed the consistency and convergence of a derived fourth-stage second-order explicit Runge Kutta method that is capable of handling initial value problems.

Runge-kutta methods are numerical (one-step) methods for solving initial value problems of the form:

$$y'(x) = f(x,y), \quad y(x_0) = y_0. \qquad (1.1)$$

Also, according to Agbeboh (2006), Thomas et al (2001), and Yakubu (2010), in Ordinary Differential Equations, initial value problems are problems with subsidiary conditions which are called initial conditions and are applicable to solving real life problems. This can be used to analyze growth and decay problems in real life situations.

In the works of Turker (1980), and William (2002), Explicit Runge-Kutta methods have proven to be one of the best methods for solving initial value problems in Ordinary Differential Equations. However, the method is subject to improvement, hence more research is still been carried out to get better efficiency and accuracy of the method. Many researchers have worked to improve on the accuracy of the method as can been seen in the work of Agbeboh (2013, 2014 and 2015) and Van der Houwen (2015). More recent works are those of Brugnano et al (2019), Barletti et al (2020) and Gianluca et al (2021).

2.0 METHOD OF DERIVATION

i. Obtaining the Taylor series expansion of $k_{i's}^{;}$ about the point (x_n, y_n), i = 2,3,4,
ii. Carry out substitution to ensure that all the $k_{i's}^{;}$ are in terms of k_1 only.
iii. Insert the $k_{i's}^{;}$ in terms of k_1 only into $b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$
iv. Compare the coefficients with Taylor series expansion involving f of the form:

$$\phi(x,y,h) = f + \frac{h}{2!}f_x + \frac{h^2}{3!}\left(f_{xx} + 2ff_{xy} + f_xf_y\right) + \frac{h^3}{4!}(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + 3f_xf_{xy} + 5ff_yf_{xy} + 3ff_xf_{yy} + f_{xx}f_y + f_xf_y^2)$$

v. As a result, also, a set of linear/non-linear equations will be generated.
vi. Vary the parameters from the set of equations generated above. A new fourth-stage second-order explicit Runge-Kutta formula will be derived.

3.0 DERIVATION OF THE METHOD

From the general Runge-Kutta scheme, we have the second-order fourth-stage method below:

$$y_i = y_{i-1} + h_i(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4)$$
$$k_1 = f(x_{i-1}, y_{i-1})$$
$$k_2 = f(x_{i-1} + c_2h_i, y_{i-1} + h_i(a_{21}k_1))$$
$$k_3 = f(x_{i-1} + c_3h_i, y_{i-1} + h_i(a_{31}k_1 + a_{32}k_2))$$
$$k_4 = f(x_{i-1} + c_4h_i, y_{i-1} + h_i(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)) \qquad (3.1)$$

Using Taylor's Series Expansion for $k_{i's}h_i = h$

$$k_1 = f(x_{i-1}, y_{i-1}) = f$$
$$k_2 = \sum_{r=0}^{\infty}\frac{1}{r!}(c_2h\frac{\partial}{\partial x} + ha_{21}k_1\frac{\partial}{\partial y})^r f$$
$$k_3 = \sum_{r=0}^{\infty}\frac{1}{r!}(c_3h\frac{\partial}{\partial x} + h(a_{31}k_1 + a_{32}k_2)\frac{\partial}{\partial y})^r f$$
$$k_4 = \sum_{r=0}^{\infty}\frac{1}{r!}(c_4h\frac{\partial}{\partial x} + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\frac{\partial}{\partial y})^r f \qquad (3.2)$$

Expanding we have:

$$k_1 = f$$
$$k_2 = f + \left(c_2hf_x + ha_{21}k_1f_y\right) + \frac{1}{2!}(c_2hf_x + ha_{21}k_1f_y)^2 + (0h^3)$$
$$k_3 = f + \left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)f_y\right) + \frac{1}{2!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)f_y\right)^2 + (0h^3)$$
$$k_4 = f + \left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)f_y\right) + \frac{1}{2!}(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)f_y)^2 + (0h^3) \qquad (3.3)$$

Now expanding the $k_{i's}$ in terms of f only:

$$k_1 = f$$
$$k_2 = f + h\left(c_2f_x + a_{21}ff_y\right) + \frac{h^2}{2!}\left(c_2^2f_{xx} + 2c_2a_{21}ff_{xy} + a_{21}^2f^2f_{yy}\right) + 0(h^3)k_3 = f + h\left(c_3f_x + (a_{31} + a_{32})ff_y\right) + \frac{h^2}{2!}\left(c_3^2f_{xx} + 2c_3(a_{31} + a_{32})ff_{xy} + (a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{yy} + 2c_2a_{32}f_xf_y + 2a_{31}a_{32}ff_y^2\right) + 0(h^3)k_4 = f + h\left(c_4f_x + (a_{41} + a_{42} + a_{43})ff_y\right) + \frac{h^2}{2!}(c_4^2f_{xx} + (2c_2a_{42} + 2c_3a_{43})f_xf_y + (2c_4a_{41} + 2c_4a_{42} + 2c_4a_{43})ff_{xy} +$$

$(2a_{21}a_{42} + a_{31}a_{43} + 2a_{32}a_{43})ff_y^2 + (2a_{41}^2 + 4a_{41}a_{42} + 4a_{41}a_{43} + 2a_{42}^2 + 4a_{42}a_{43} + 2a_{43}^2)f^2 f_{yy}) + 0(h^3)$ (3.4)

But$\emptyset(x,y,h) = \sum_{r=1}^{4} b_r k_r = (b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4)\emptyset(x,y,h) = (b_1 f + b_2\Big(f + h(c_2 f_x + a_{21}ff_y) + \frac{h^2}{2!}(c_2^2 f_{xx} + 2c_2a_{21}ff_{xy} + a_{21}^2 f^2 f_{yy})\Big) + b_3(f + h(c_3f_x + (a_{31} + a_{32})ff_y) + \frac{h^2}{2!}(c_3^2 f_{xx} + 2c_3(a_{31} + a_{32})ff_{xy} + (a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2 f_{yy} + 2c_2a_{32}f_xf_y + 2a_{31}a_{32}ff_y^2)) + b_4(f + h(c_4f_x + (a_{41} + a_{42} + a_{43})ff_y) + \frac{h^2}{2!}(c_4^2 f_{xx} + (2c_2a_{42} + 2c_3a_{43})f_xf_y + (2c_4a_{41} + 2c_4a_{42} + 2c_4a_{43})ff_{xy} + (2a_{21}a_{42} + a_{31}a_{43} + 2a_{32}a_{43})ff_y^2 + (2a_{41}^2 + 4a_{41}a_{42} + 4a_{41}a_{43} + 2a_{42}^2 + 4a_{42}a_{43} + 2a_{43}^2)f^2 f_{yy})))$ (3.5)

The Taylor Series Expansion is:

$\emptyset_T(x,y,h) = f + \frac{1}{2}h(f_x + ff_y) + \frac{1}{6}h(f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_xf_y + ff_y^2) + 0(h^3)$ (3.6)

Equating (3.5) with (3.6), we have the Equations below:

$b_1 + b_2 + b_3 + b_4 = 1$ (3.7)

$b_2c_2 + b_3c_3 + b_4c_4 = \frac{1}{2}$ (3.8)

$b_2a_{21} + b_3(a_{31} + a_{32}) + b_4(a_{41} + a_{42} + a_{43}) = \frac{1}{2}$ (3.9)

$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = \frac{1}{3}$ (3.10)

$b_2c_2a_{21} + b_3c_3(a_{31} + a_{32}) + b_4c_4(a_{41} + a_{42} + a_{43}) = \frac{2}{3}$ (3.11)

$b_2c_{21}^2 + b_3(a_{31} + a_{32})^2 + 2b_4(a_{41} + a_{42} + a_{43})^2 = \frac{1}{3}$ (3.12)

$2b_3c_2a_{32} + 2b_4c_2a_{42} + 2b_4c_3a_{43} = \frac{1}{3}$ (3.13)

$2b_3a_{21}a_{32} + 2b_4a_{21}a_{42} + 2b_4a_{43}(a_{31} + a_{32}) = \frac{1}{3}$ (3.14)

Solving the above Eight (8) equations,

Set $c_1 = 0, \ c_4 = 1, c_2 = \frac{1}{4}, \ c_3 = \frac{3}{4},$ (3.15)

The eight (8) Equations become:

$b_1 + b_2 + b_3 + b_4 = 1$

$b_2 + 3b_3 + 4b_4 = 2$ (3.16)

$b_2a_{21} + b_3(a_{31} + a_{32}) + b_4(a_{41} + a_{42} + a_{43}) = \frac{1}{2}$ (3.17)

$b_2a_{21} + 3b_3(a_{31} + a_{32}) + 4b_4(a_{41} + a_{42} + a_{43}) = \frac{8}{3}$ (3.18)

$b_2 + 9b_3 + 16b_4 = \frac{16}{3}$ (3.19)

$b_2a_{21}^2 + b_3(a_{31} + a_{32})^2 + 2b_4(a_{41} + a_{42} + a_{43})^2 = \frac{1}{3}$ (3.20)

$b_3a_{32} + b_4a_{42} + 3b_4a_{42} = \frac{2}{3}$ (3.21)

$b_3a_{31}a_{32} + b_4a_{21}a_{42} + b_4a_{43}(a_{31}a_{32}) = \frac{1}{6}$ (3.22)

Solving (3.1.7), (3.1.16) and (3.1.19), we have:

$b_1 = \frac{1}{4}, \quad b_2 = \frac{1}{18}, \ b_3 = \frac{5}{6}, \quad b_4 = -\frac{5}{36}$ (3.23)

Letting $a_{21} = A, a_{31} + a_{32} = B, a_{41} + a_{42} + a_{43} = C$ (3.24)

Putting (3.15), (3.23) and (3.24) into the remaining Equations, we have:

$a_{21} = \frac{3}{28}, \quad a_{31} = -\frac{283}{140}, \quad a_{32} = \frac{6}{5}, \quad a_{41} = -\frac{339}{35}, \quad a_{42} = \frac{3}{5}, \quad a_{43} = \frac{3}{5}$ (3.25)

Putting (3.15), (3.23) and (3.25) into (3.1), we have the Second-Order Fourth-Stage formula below:

$y_i = y_{i-1} + \frac{h_i}{36}(9k_1 + 2k_2 + 30k_3 - 5k_4)$

$k_1 = f(x_{i-1}, y_{i-1})$

$k_2 = f\left(x_{i-1} + \frac{h_i}{4}, \quad y_{i-1} + \frac{3h_i}{28}k_1\right)$

$k_3 = f\left(x_{i-1} + \frac{3h_i}{4}, y_{i-1} + \frac{h_i}{140}(-283k_1 + 168k_2)\right)$

$k_4 = f\left(x_{i-1} + h_i, y_{i-1} + \frac{h_i}{35}(-339k_1 + 21k_2 + 21k_3)\right)$

4.0 CONSISTENCY AND CONVERGENCE ANALYSIS OF THE FOURTH STAGE SECOND ORDER EXPLICIT RUNGE KUTTA METHOD

Theorem 1: The explicit fourth-stage fourth-order method is consistent if it converges to the initial value problem

$$y' = f(x,y), y(x_o) = y_o.$$

Proof:Using the exact solution $y(x_n)$ of the initial value problem:
$y' = f(x,y), y(x_o) = y_o$, we have that:

$$T_n(h^3) = y_{n+1} - y_n$$

$$= \frac{h}{36}\Bigg(9f(x_n, y_n) + 2f\left(x_n + \frac{h}{4}, y_n + \frac{3h}{28} f(x_n, y_n)\right)$$

$$+ 30 f\left(x_n + \frac{3h}{4}, \quad y_n + \frac{h}{140}\left(-283 f(x_n, y_n) + 168 f\left(x_n + \frac{h}{4}, \quad y_n + \frac{3h}{28} f(x_n, y_n)\right)\right)\right)$$

$$- 5 f(x_n + h,$$

$$y_n + \frac{h}{35}(-339f(x_n, y_n) + 21f\left(x_n + \frac{h}{4}, \quad y_n + \frac{3h}{28} f(x_n, y_n)\right)$$

$$+ 21f\left(x_n + \frac{3h}{4}, y_n + \frac{h}{140}\left(-283f(x_n, y_n) + 168f\left(x_n + \frac{h}{4}, \quad y_n + \frac{3h}{28} f(x_n, y_n)\right)\right)\right)$$

Dividing all through by h and taking the limit of both sides as $h \to 0$, *we have*:

$$T_n(h^3) = \frac{y_{n-1} - y_n}{h} = \frac{1}{36}\left[9f(x_n, y_n) + 2f(x_n, y_n) + 30f(x_n, y_n) - 5f(x_n, y_n)\right]$$

$= \frac{1}{36}[36f(x_n, y_n)] = f(x_n, y_n)$ which is the initial value problem.

$y' = f(x_n, y_n), \ y(x_0) = y_0.$

Hence our method is consistent and convergent .

4.0CONCLUSION

It is clearly seen from the analyses above that the method converges to the initial value problem. Hence, the method is consistent. As such, it will be consistent and convergent in handling initial value problems in ordinary differential equations. These are necessary properties any numerical method should possess.

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