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ROOTED TREE ANALYSIS OF AN EXPLICIT FOURTH-STAGE FOURTH ORDER RUNGE-KUTTA METHOD

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ABTRACT

This research paper is aimed at using Butcher's rooted trees to separate the f(y) functional derivatives from the f(x,y) functional derivatives after applying Taylor series expansion on the general fourth stage fourth order Runge Kutta method. This approach revealed that the f(y) functional derivatives generated a set of linear/ nonlinear equations that gave birth to a fourth stage fourth order Runge Kutta formula. This idea is derivable from general graphs and combinatorics.

Keywords: *Rooted tree diagram, Comparison, Variation, explicit, f(y) functional derivatives, f(x,y) functional derivatives Runge-Kutta Method*

INTRODUCTION

In mathematics, a rooted tree is a directed graph with a distinguished vertex or node called the root which determines the direction of other vertices or nodes in the tree. According to Williams (2002), a rooted tree is a collection of nodes connected by directed edges with no circuits having a starting vertex or node called the root. It is a nonlinear data structure, compared to arrays, linked lists, stacks and queues which are linear data structures. A rooted tree can be empty with no nodes or consisting of one node called the root and zero or one or more sub trees. A rooted tree has the following general properties: One node is distinguished as a root; every node or vertex (exclude a root) is connected by a directed edge from one node to another node; that is direction from a parent to children.

According to Tucker (1980), a rooted tree is a graph with a designated vertex called a root such that there is a unique path from the root to any other vertex or node in the tree. The length of the unique path from a vertex v to the root is called the level number of vertex v. The root has level number 0. For any vertex v (except the root), the father of v is the unique vertex v' with an edge common with v and a smaller level number. Conversely, v is a son of v'. A vertex with no sons is a leaf. Rooted trees are sometimes called pointed trees. The tree with one vertex is the “tree” with zero vertex is designated by 1. An important operation is the merging of trees. If $t_1, \; t_2, \dots t_m$ are trees, $t = B^{+}(t_1, \; t_2, \dots t_m)$ is defined as the tree obtain by creating a new vertex r and by joining the root of $t_1, \; t_2, \dots t_m$ to r, which becomes the root of t. This operation is also denoted by $t = [t_1, \; t_2, \dots t_m]$.

According to Thomas et al (2001), rooted trees are single rooted connected graphs with no cycles and are used in the analysis of order for general linear methods. According to William (2002), T denotes the set of all rooted trees including the empty set. For $t \epsilon T$, the order of the tree denoted $r(t)$, is the number of vertices on the tree. All trees subject to r(t) >1 can be represented recursively by deleting the root of t and denoting the distinct subtrees left as $t_1, \; t_2, \dots t_m$. The relationship between t and $t_1, \; t_2, \dots t_m$, is denoted by

t = $[t_1^{n1}, \; t_2^{n2}, \dots t_m^{nm}]$, where $n_1, \; n_2, \dots n_m$ denote the number of times $t_1, \; t_2, \dots t_m$ respectively occur. This is known as grafting the trees $t_1, \; t_2, \dots t_m$ to a new root”.

In the work of Butcher (2009, 2010), the first few derivatives of the problem:

$y'(x) = f\big(y(x)\big),$

we can rewrite the equation in the form:

$y'(x) = f_i\big(y_1(x), y_2(x), \dots, y_N(x)\big), \quad i = 1,2,\dots,N,$ using the chain rule, we have:

$y_i'' = \sum_{j=1}^{N} \frac{\partial f_i}{\partial y_i} y_j' = \sum_{j=1}^{N} \frac{\partial f_i}{\partial y_i} f_j$

This can be written compactly as: $y''(x) = f'f,$

Where f is the vector $f\big(y(x)\big)$ and f' is the linear operator defined by the matrix of partial derivatives.

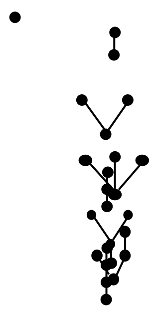
Introducing the bi-linear operator f'' of second partial derivates and the third derivate of y is found to be: $y'''(x) = f''(f,f) + f'f'f$

$y^{(iv)}(x) = f'''(f,f,f) + f''(f',f,f) + f'f''(f,f) + f'f'f'f$

Expressions like $f, \; f'f, \; f''(f,f), \; f'f'f$ are known as “elementary differentials.” They are related to rooted trees by the very simple rule that $f^{(m)}$ is represented by a parent of m children.

Figure 1 The various elementary differentials and their rooted trees

f is represented by
$f'f$ is represented by
$f''(f,f)$ is represented by
$f'f'f$ is represented by
$f'''(f,f,f)$ is represented by
$f''(f',f,f)$ is represented by
$f'f''(f,f)$ is represented by
$f'f'f'f$ is represented by



METHOD OF DERIVATION

The following the procedures are adopted before applying Butchers rooted trees to get the equations:
Given the general Runge-Kutta method below:

$y_{n+1} = y_n + h\,\phi(x_n, y_n, h)$

$\phi(x_n, y_n, h) = \sum_{r=1}^{R} b_r\, k_r$

$k_1 = f(x, y)$

$$k_r = f(x + hc_r, y + h \sum_{s=1}^{r-1} a_{rs} k_s), r = 2, 3, \dots R, \quad (2.1)$$

We derive an explicit Fourth-Stage Fourth-order method by:

i. Obtaining the Taylor series expansion of $k_{i's}$ about the point (x_n, y_n), i=2,3,4,

ii. Carry out substitution to ensure that all the $k_{i's}^{\prime}$ are in terms of k_1 only.

iii. Insert the $k_{i's}^{\prime}$ in terms of k_1 only into $b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$

iv. Separate all functional derivatives involving only f(y) with their coefficient from all functional derivatives involving f(x,y) and their coefficients.

v. Compare the coefficients of all functional derivatives involving only f(y) with Taylor series expansion involving only partial derivatives with respect to y of the form:

$$\emptyset_T(x,y,h) = f + \frac{h}{2!}ff_y + \frac{h^2}{3!}(ff_y^2 + f^2f_{yy}) + \frac{h^3}{4!}(4f^2f_yf_{yy} + ff_y^3 + f^3f_{yyy}) + \frac{h^4}{5!}(7f^3f_yf_{yyy} + 4f^3f_{yy}^2 + 11f^2f_y^2f_{yy} + ff_y^4 + f^4f_{yyyy})$$

vi. Arising from (v), a set of linear/ nonlinear equations will be generated. Represent these equations and their functional derivatives on Butcher's rooted trees.

rom the scheme in (2.1), the fourth-stage fourth–order method is:

$$y_{n+1} = y_n + h\,(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4)$$
$$k_1 = f\,(x_n, y_n)$$
$$k_2 = f\,(x_n + c_2, y_n + ha_{21}k_1)$$
$$k_3 = f\,(x_n + c_3h, y_n + h(a_{31}k_1 + a_{32}k_2))$$
$$k_4 = f\,(x_n + c_4h, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)) \quad (2.2)$$

Using Taylor's series expansion for $k_i's$, we have:

$$k_1 = f\,(x_n, y_n)$$
$$k_2 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_2h\frac{\partial}{\partial x} + ha_{21}k_1\frac{\partial}{\partial y})^r f(x_n, y_n)$$
$$k_3 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_3h\frac{\partial}{\partial x} + h(a_{31}k_1 + a_{32}k_2)\frac{\partial}{\partial y})^r f(x_n, y_n)$$
$$k_4 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_4h\frac{\partial}{\partial x} + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\frac{\partial}{\partial y})^r f(x_n, y_n) \quad (2.3)$$

Hence, we have:

$$k_1 = f$$
$$k_2 = f + (c_2hf_x + ha_{21}k_1f_y) + \frac{1}{2!}(c_2hf_x + ha_{21}k_1f_y)^2 + \frac{1}{3!}(c_2hf_x + ha_{21}k_1f_y)^3 + \frac{1}{4!}(c_2hf_x + ha_{21}k_1f_y)^4 + 0(h^5)$$
$$k_3 = f + (c_3hf_x + h(a_{31}k_1 + a_{32}k_2)f_y) + \frac{1}{2!}(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)f_y)^2 + \frac{1}{3!}(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)f_y)^3 + \frac{1}{4!}(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)f_y)^4 + 0(h^5)$$
$$k_4 = f + (c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)f_y) + \frac{1}{2!}(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)f_y)^2 + \frac{1}{3!}(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)f_y)^3 + \frac{1}{4!}(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)f_y)^4 + 0(h^5)$$

(2.4)

Expanding fully and substituting the various k_i' s, i = 2, 3, 4 into their various positions in terms of k_1 only and collecting like terms, in terms of y derivatives and (x, y) derivatives separately, we have:

$$k_1 = f$$

$$k_2 = f + ha_{21}ff_y + \frac{h^2}{2!}a_{21}^2f^2f_{yy} + \frac{h^3}{3!}a_{21}^3f^3f_{yyy} + \frac{h^4}{4!}a_{21}^4f^4f_{yyyy} + hc_2f_x$$
$$+ \frac{h^2}{2!}c_2^2f_{xx} + h^2c_2a_{21}ff_{xy}\frac{h^3}{3!}c_2^3f_{xxx} + \frac{h^3}{2!}c_2^2a_{21}ff_{xxy} + \frac{h^3}{2!}c_2a_{21}^2f^2f_{xyy}$$
$$+ \frac{h^4}{4!}c_2^4f_{xxxx} + \frac{h^4}{3!}c_2^3a_{21}ff_{xxxy}\frac{h^4}{2!2!}c_2^2a_{21}^2f^2f_{xxyy} + \frac{h^4}{3!}c_2a_{21}^3f^3f_{xyyy} + 0\,(h^5)$$

$$k_3 = f + h\,(a_{31} + a_{32})ffy + h^2a_{21}a_{32}ff_y^2 + \frac{h^2}{2!}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{yy} + \frac{h^3}{3!}a_{21}a_{32}(a_{21} + 2(a_{31} + a_{32}))f^2f_yf_{yy} + \frac{h^3}{3!}(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3f_{yyy} + \frac{h^4}{3!}(a_{32}a_{21}^3 + 3a_{31}^2a_{32}a_{21} + 3a_{32}^3a_{21} + 6a_{31}a_{32}^2a_{21})f^3f_yf_{yyy} + \frac{h^4}{2!}a_{21}^2a_{32}(a_{31} + a_{32})f^3f_{yy}^2 + \frac{h^4}{2!}a_{32}^2a_{21}^2f^2f_y^2f_{yy} + \frac{h^4}{4!}(a_{31}^4 + 4a_{31}^3a_{32} + 6a_{31}^2a_{32}^2 + 4a_{31}a_{32}^3 + a_{32}^4)f^4f_{yyyy} + hc_3f_x + \frac{h^2}{2!}c_3^2f_{xx} + h^2c_3(a_{31} + a_{32})ff_{xy} + h^2c_2a_{32}f_xf_y + \frac{h^3}{3!}c_3^3f_{xxx} + \frac{h^3}{2!}c_3^2(a_{31} + a_{32})ff_{xxy} + \frac{h^3}{2!}c_3(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{xyy} + h^3c_2a_{32}(a_{31} + a_{32})ff_xf_{yy} + h^3a_{21}a_{32}(c_2 + c_3)ff_yf_{xy} + \frac{h^3}{2!}c_2^2a_{32}f_yf_{xx} + h^3c_2c_3a_{32}f_xf_{xy} + \frac{h^4}{4!}c_3^4f_{xxxx} + \frac{h^4}{3!}c_2^3a_{32}f_{xxx}f_y + \frac{h^4}{2!}c_3^2c_2a_{32}f_xf_{xxy} + \frac{h^4}{2!}a_{21}a_{32}(c_2^2 + c_3^2)ff_yf_{xxy}\frac{h^4}{3!}a_{21}a_{32}(2c_2a_{31} + 3c_2a_{21} + 6c_3a_{31})f^2f_yf_{xyy}$$

$$+ \frac{h^4}{2!} c_3 a_{32} c_2^2 f_{xx} f_{xy} + \frac{h^4}{2!} c_2^2 a_{32}(a_{31} + a_{32}) f f_{xx} f_{yy} + \frac{h^4}{2!} a_{21} a_{32}(2c_2 a_{31} + 2c_2 a_{32} + c_3 a_{21}) f^2 f_{xy} f_{yy}$$
$$+ h^4 c_3 a_{32} c_2 a_{21} f f_{xy}^2 + \frac{h^4}{2!} a_{32}^2 c_2^2 f_x^2 f_{yy} + h^4 a_{32}^2 a_{21} c_2 f f_x f_y f_{yy}$$
$$+ \frac{h^4}{2!} c_3 c_2 a_{32}(6a_{31} + 2a_{32}) f f_x f_{xyy} + \frac{h^4}{2!} c_2 a_{32}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2) f^2 f_x f_{yyy}$$
$$+ \frac{h^4}{3!} c_3^3 (a_{31} + a_{32}) f f_{xxxy} + \frac{h^4}{2!\,2!} c_3^2 (a_{31}^2 + 2a_{31}a_{32} + a_{32}^2) f^2 f_{xxyy}$$
$$+ \frac{h^4}{3!} c_3 (a_{31}^3 + 3a_{31}^2 a_{32} + 3a_{31}a_{32}^2 + a_{32}^3) f^3 f_{xyyy} + 0\,(h^5)$$

$$k_4 = f + h\,(a_{41} + a_{42} + a_{43}) f f_y + h^2 (a_{21}a_{42} + a_{31}a_{43} + a_{32}a_{43}) f f_y^2 +$$

$$\frac{h^2}{2!}(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + 2a_{42}a_{43} + a_{42}^2 + a_{43}^2) f^2 f_{yy}$$

$$+ \frac{h^3}{2!}(a_{21}^2 a_{42} + a_{31}^2 a_{43} + 2a_{31}a_{32}\,a_{43} + a_{32}^2 a_{43} + 2a_{21}a_{41}\,a_{42}$$

$$+ 2a_{31}a_{41}\,a_{43} + 2a_{32}a_{41}\,a_{43} + 2a_{31}a_{42}\,a_{43} + 2a_{32}a_{42}\,a_{43} + 2a_{21}a_{42}\,a_{43}$$

$$+ 2a_{21}a_{42}^2 + 2a_{31}a_{43}^2 + 2a_{32}a_{43}^2)\, f^2 f_y f_{yy} + h^3 a_{21}a_{32}\,a_{43} f f_y^3 + \frac{h^3}{2!}(a_{41}^3 + 3a_{41}^2 a_{42} + 3a_{41}^2 a_{43} + 3a_{41}a_{42}^2$$
$$+ 6a_{41}a_{42}a_{43} + 3a_{42}^2 a_{43} + 3a_{41}a_{43}^2 + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3) f^3 f_{yyy} + \frac{h^4}{3!}(a_{31}^3 a_{43}$$
$$+ 3a_{31}^2 a_{32}a_{43} + 3a_{31}a_{32}^2 a_{43} + 3a_{21}a_{41}^2 a_{42} + 3a_{31}a_{41}^2 a_{43} + 3a_{32}a_{41}^2 a_{43} + 6a_{21}a_{41}a_{42}^2$$
$$+ 6a_{31}a_{41}a_{42}a_{43} + 6a_{32}a_{41}a_{42}a_{43} + 6a_{21}a_{41}a_{42}a_{43} + 3a_{31}a_{41}^2 a_{43} + a_{42}a_{21}^3 + 3a_{32}a_{42}^2 a_{43}$$
$$+ 6a_{21}a_{42}^2 a_{43} + 6a_{31}a_{41}a_{43}^2 + 6a_{32}a_{41}a_{43}^2 + 6a_{31}a_{42}a_{43}^2 + 6a_{32}a_{42}a_{43}^2 + 3a_{21}a_{42}a_{43}^2$$
$$+ 3a_{21}a_{43}^3 + 3a_{31}a_{43}^3 + 3a_{32}a_{43}^3) + \frac{h^4}{2!}(a_{21}^2 a_{32}a_{43} + 2a_{21}a_{31}a_{32}a_{43} + 2a_{21}a_{32}^2 a_{43}$$
$$+ 2a_{21}a_{32}a_{41}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + 2a_{21}a_{31}a_{42}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + a_{21}^2 a_{42}^2 + a_{21}a_{32}a_{43}^2 +$$

$$a_{31}^2 a_{43}^2 + 2a_{31}a_{32}a_{43}^2 + a_{32}^2 a_{43}^2) f^2 f_y^2 f_{yy} + \frac{h^4}{2!}(a_{21}^2 a_{41}a_{42} + a_{31}^2 a_{41}a_{42} +$$

$2a_{31}a_{32}a_{41}a_{43} + a_{32}^2 a_{41}a_{43} + a_{31}^2 a_{41}a_{43} + 2a_{31}a_{32}a_{42}a_{43} + a_{32}^2 a_{42}a_{43} + \frac{a_{21}^2 a_{42}^2}{2!} + \frac{a_{31}^2 a_{43}^2}{2!} + a_{31}a_{32}a_{43}^2 +$
$\frac{a_{31}^2 a_{43}^2}{2!}) f^3 f_{yy}^2 + \frac{h^4}{4!}\,(a_{41}^4 + 4_{41}^3 a_{42} + 4a_{41}^3 a_{43} + 6a_{41}^2 a_{41}^2 + 12a_{41}^2 a_{42}a_{43} + 6a_{42}^2 a_{43}^2 + 4a_{41}a_{43}^3 + 4a_{42}a_{43}^3 +$
$12a_{41}a_{42}^2 a_{43} + 2a_{41}a_{42}a_{43}^2 + 6a_{41}^2 a_{43}^2 + 4a_{42}^3 a_{43} + 4a_{41}a_{42}^3 + a_{42}^4 + a_{43}^4)\, f^4 f_{yyyy} + \mathrm{h}c_4 f_x + h^2(c_4 a_{42} + c_3 a_{43}) f_x f_y$
$+ \frac{h^2}{2!} c_4^2 f_{xx} + h^2 c_4 (a_{41} + a_{42} + a_{43}) f f_{xy} + \frac{h^3}{2!}\,(c_2^2 a_{42} + c_3^2 a_{43}) f_{xx} f_y + h^2(c_4 a_{21}a_{42} + c_3 a_{31}a_{43} + c_3 a_{32}a_{43}) f f_{xy} f_y +$
$h^3 c_2 a_{32}a_{43} f_x f_y^2 + h^3(c_2 c_4 a_{42} + c_3 c_4 a_{43})\, f_x f_{xy} + h^3 c_2 a_{32}a_{43} f_x f_y^2 + h^3(c_2 c_4 a_{42} + c_3 c_4 a_{43})\, f_x f_{xy} + h^3(c_2 a_{21}a_{42} +$
$c_4 a_{31}a_{43} + c_4 a_{32}a_{43}) f f_y f_{xy} + h^3(c_2 a_{41}a_{42} + c_3 a_{41}a_{43} + c_3 a_{42}a_{43} + c_2 a_{42}a_{43} + c_2 a_{42}^2 + c_3 a_{43}^2) f f_x f_{yy} + \frac{h^3}{3!} c_4^3 f_{xxx} +$
$\frac{h^3}{2!}\,(c_4^2 a_{41} + c_4^2 a_{42} + c_4^2 a_{43}) f f_{xxy} + \frac{h^3}{2!} c_4 (a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + a_{42}^2 + 2a_{42}a_{43} + a_{43}^2) f^2 f_{xyy} +$
$\frac{h^4}{3!}\,(c_2^3 a_{42} + c_3^3 a_{43}) f_{xxx} f_y + \frac{4}{3!}(3c_2^2 a_{21}a_{42} + c_3^2 a_{31}a_{43} + 3c_3^2 a_{32}a_{43} + 3c_4^2 a_{21}a_{42} + 3c_4^2 a_{31}a_{43} + 3c_4^2 a_{32}a_{43}) f f_{xxy} f_y + \frac{h^4}{2!}$
$(c_2 a_{21}^2 a_{42} + 2c_3 a_{31}a_{32}a_{43} + c_3 a_{31}^2 a_{43} + c_3 a_{32}^2 a_{43} +$

$2c_4 a_{21}a_{41}a_{42} + 2c_4 a_{31}a_{41}a_{43} + 2c_4 a_{32}a_{41}a_{43} + 2c_4 a_{21}a_{42}^2 + 2c_4 a_{31}a_{42}a_{43} +$

$2c_4 a_{32}a_{42}a_{43} + 2c_4 a_{21}a_{42}a_{43} + 2c_4 a_{31}a_{43}^2 + 2c_4 a_{32}a_{43}^2) f^2 f_y f_{xyy} +$

$\frac{h^4}{3!}\,(c_2^2 a_{32}a_{43}) f_{xx} f_y^2 + h^4(c_2 a_{21}a_{32}a_{43} + c_3 a_{21}a_{32}a_{43} + c_4 a_{21}a_{32}a_{43}) f f_{xy} f_y^2 +$

$\frac{h^4}{2!}(2c_2 a_{31}a_{32}a_{43} + 2c_2 a_{32}^2 a_{43} + 2c_2 a_{32}a_{41}a_{43} + 2c_2 a_{32}a_{42}a_{43} + 2c_2 a_{31}a_{42}a_{43} + 2c_2 a_{32}a_{42}a_{43} + 2c_3 a_{21}a_{42}a_{43} +$
$2c_2 a_{21}a_{42}^2 + c_2 a_{32}a_{43}^2 + c_3 a_{31}a_{43}^2 + c_3 a_{32}a_{43}^2 + c_3 a_{31}a_{43}^2 + c_3 a_{32}a_{43}^2) f f_x f_y f_{yy} + h^4(c_2 c_3 a_{32}a_{43} + c_2 c_4 a_{32}a_{43}) f_x f_y f_{xy}$
$+ \frac{h^4}{2!}\,(c_2^2 c_4 a_{42} + c_3^2 c_4 a_{43}) f_{xx} f_{xy} + h^4(c_2 c_4 a_{21}a_{42} + c_3 c_4 a_{31}a_{43} + c_3 c_4 a_{32}a_{43}) f f_{xy}^2 + \frac{h^4}{2!}\,(c_4 a_{21}^2 a_{42} + c_4 a_{31}^2 a_{43} +$
$2c_4 a_{31}a_{32}a_{43} + c_4 a_{32}^2 a_{43} + 2c_2 a_{21}a_{41}a_{42} + 2c_3 a_{31}a_{41}a_{43} + 2c_3 a_{32}a_{41}a_{43} + 2c_3 a_{31}a_{42}a_{43} + 2c_3 a_{32}a_{42}a_{43} + c_2 a_{21}a_{42}^2 +$
$c_3 a_{31}a_{43}^2 + c_3 a_{32}a_{43}^2)\; f^2 f_{xy} f_{yy} + \frac{h^4}{2!}(2c_2 c_3 a_{42}a_{43} + c_2^2 a_{42}^2 + c_3^2 a_{43}^2) f_x^2 f_{yy} + \frac{h^4}{2!}\,(c_2 c_4^2 a_{42} + c_3 c_4^2 a_{43}) f_x f_{xxy} +$

$h^4(c_2c_4a_{41}a_{42} + c_3c_4a_{41}a_{43} + c_2c_4a_{42}^2 + c_3c_4a_{42}a_{43}) + c_2c_4a_{42}a_{43} + c_3c_4a_{43}^2)ff_xf_{xyy} + \frac{h^4}{2!}(c_2c_{41}^2a_{42} + c_3c_{41}^2a_{43} + 2c_2a_{41}a_{42}^2 + 2c_3a_{41}a_{42}a_{43} + 2c_2a_{41}a_{42}a_{43} + c_3c_{42}^2a_{43} + 2c_2c_{42}^2a_{43} + 2c_3a_{41}a_{43}^2 + 2c_3a_{42}a_{43}^2 + c_2a_{42}a_{43}^2 + c_2c_{42}^3 + c_3a_{43}^3)f^2f_xf_{yyy} + \frac{h^4}{4!}c_4^4f_{xxxx} + \frac{h^4}{3!}(c_4^3a_{41} + c_4^3a_{42} + c_4^3a_{43})ff_{xxxy} + \frac{h^4}{2!2!}c_4^2(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + a_{42}^2 + 2a_{42}a_{43} + a_{43}^2)f^2f_{xxyy} + \frac{h^4}{3!}c_4(a_{41}^3 + 3a_{41}^2a_{42} + 3a_{41}^2a_{43} + 3a_{41}a_{42}^2 + 6a_{41}a_{42}a_{43} + 3a_{42}^2a_{43} + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3) f^3f_{xyyy} + \frac{h^4}{2!2!}(2c_2^2a_{41}a_{42} + 2c_3^2a_{41}a_{43} + 2c_3^2a_{42}a_{43} + c_2^2a_{42}^2 + c_3^2a_{43}^2)ff_{xx}f_{yy.} + 0(h^5).$

(2.5)

Putting the $k_{i's}^{;}$ (y derivatives only) into $y_{n+1} = y_n + h(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$ *where* $\phi(x,y,h) = b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$*, we have:*

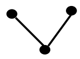
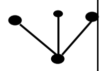
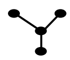
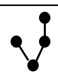
$$y_{n+1} = y_n + h(b_1f + b_2(f + ha_{21}ff_y + \frac{h^2}{2!}a_{21}^2f^2f_{yy} + \frac{h^3}{3!}a_{21}^3f^3f_{yyy} + \frac{h^4}{4!}a_{21}^4f^4f_{yyyy}) + b_3(f$$
$$+ h(a_{31} + a_{32})ffy + h^2a_{21}a_{32}ff_y^2 + \frac{h^2}{2!}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{yy}$$
$$+ \frac{h^3}{3!}a_{21}a_{32}(a_{21} + 2(a_{31} + a_{32}))f^2f_yf_{yy} + \frac{h^3}{3!}(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3f_{yyy}$$
$$+ \frac{h^4}{3!}(a_{32}a_{21}^3 + 3a_{31}^2a_{32}a_{21} + 3a_{32}^3a_{21} + 6a_{31}a_{32}^2a_{21})f^3f_yf_{yyy} + \frac{h^4}{2!}a_{21}^2a_{32}(a_{31} + a_{32})f^3f_{yy}^2$$
$$+ \frac{h^4}{2!}a_{32}^2a_{21}^2f^2f_y^2f_{yy} + \frac{h^4}{4!}(a_{31}^4 + 4a_{31}^3a_{32} + 6a_{31}^2a_{32}^2 + 4a_{31}a_{32}^3 + a_{32}^4)f^4f_{yyyy}) + b_4(f$$
$$+ h(a_{41} + a_{42} + a_{43})ff_y + h^2(a_{21}a_{42} + a_{31}a_{43} + a_{32}a_{43})ff_y^2 +$$
$$\frac{h^2}{2!}(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + 2a_{42}a_{43} + a_{42}^2 + a_{43}^2)f^2f_{yy}$$
$$+\frac{h^3}{2!}(a_{21}^2a_{42} + a_{31}^2a_{43} + 2a_{31}a_{32}a_{43} + a_{32}^2a_{43} + 2a_{21}a_{41}a_{42}$$
$$+ 2a_{31}a_{41}a_{43} + 2a_{32}a_{41}a_{43} + 2a_{31}a_{42}a_{43} + 2a_{32}a_{42}a_{43} + 2a_{21}a_{42}a_{43}$$

$+ 2a_{21}a_{42}^2 + 2a_{31}a_{43}^2 + 2a_{32}a_{43}^2) f^2f_yf_{yy} + h^3a_{21}a_{32}a_{43}ff_y^3 + \frac{h^3}{2!}(a_{41}^3 + 3a_{41}^2a_{42} + 3a_{41}^2a_{43} + 3a_{41}a_{42}^2 + 6a_{41}a_{42}a_{43} + 3a_{42}^2a_{43} + 3a_{41}a_{43}^2 + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3)f^3f_{yyy} + \frac{h^4}{3!}(a_{31}^3a_{43} + 3a_{31}^2a_{32}a_{43} + 3a_{31}a_{32}^2a_{43} + 3a_{21}a_{41}^2a_{42} + 3a_{31}a_{41}^2a_{43} + 3a_{32}a_{41}^2a_{43} + 6a_{21}a_{41}a_{42}^2 + 6a_{31}a_{41}a_{42}a_{43} + 6a_{32}a_{41}a_{42}a_{43} + 6a_{21}a_{41}a_{42}a_{43} + 3a_{31}a_{41}^2a_{43} + a_{42}a_{21}^3 + 3a_{32}a_{42}^2a_{43} + 6a_{21}a_{42}^2a_{43} + 6a_{31}a_{41}a_{43}^2 + 6a_{32}a_{41}a_{43}^2 + 6a_{31}a_{42}a_{43}^2 + 6a_{32}a_{42}a_{43}^2 + 3a_{21}a_{42}a_{43}^2 + 3a_{21}a_{43}^3 + 3a_{31}a_{43}^3 + 3a_{32}a_{43}^3) + \frac{h^4}{2!}(a_{21}^2a_{32}a_{43} + 2a_{21}a_{31}a_{32}a_{43} + 2a_{21}a_{32}^2a_{43} + 2a_{21}a_{32}a_{41}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + 2a_{21}a_{31}a_{42}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + a_{21}^2a_{42}^2 + a_{21}a_{32}a_{43}^2 + a_{31}^2a_{43}^2 + 2a_{31}a_{32}a_{43}^2 + a_{32}^2a_{43}^2)f^2f_y^2f_{yy} + \frac{h^4}{2!}(a_{21}^2a_{41}a_{42} + a_{31}^2a_{41}a_{42} + 2a_{31}a_{32}a_{41}a_{43} + a_{32}^2a_{41}a_{43} + a_{31}^2a_{41}a_{43} + 2a_{31}a_{32}a_{42}a_{43} + a_{32}^2a_{42}a_{43} + \frac{a_{21}^2a_{42}^2}{2!} + \frac{a_{31}^2a_{43}^2}{2!} + a_{31}a_{32}a_{43}^2 + \frac{a_{31}^2a_{43}^2}{2!})f^3f_{yy}^2 + \frac{h^4}{4!}(a_{41}^4 + 4_{41}^3a_{42} + 4a_{41}^3a_{43} + 6a_{41}^2a_{41}^2 + 12a_{41}^2a_{42}a_{43} + 6a_{42}^2a_{43}^2 + 4a_{41}a_{43}^3 + 4a_{42}a_{43}^3 + 12a_{41}a_{42}^2a_{43} + 2a_{41}a_{42}a_{43}^2 + 6a_{41}^2a_{43}^2 + 4a_{42}^3a_{43} + 4a_{41}a_{42}^3 + a_{42}^4 + a_{43}^4) f^4f_{yyyy}))$ (2.6)

The Taylor series expansion for y derivatives only is:

$\emptyset_T(x,y,h) = f + \frac{h}{2!}ff_y + \frac{h^2}{3!}(ff_y^2 + f^2f_{yy}) + \frac{h^3}{4!}(4f_y^2f_yf_{yy} + ff_y^3 + f^3f_{yyy}) + \frac{h^4}{5!}(4f^3f_yf_{yyy} + 4f^3f_{yy}^2 + 11f^2f^2f_{yy} + ff_y^4 + f^4f_{yyyy})$ (2.7)

Comparing the f(y) functional derivatives with the f(y) functional derivatives from the Taylor series expansion, we have the table below having the individual equation from the individual rooted trees:

EQUATIONS	FUNCTIONAL DERIVATIVES	r(t)	TREE	ELEMENTARY DEFFRENTIALS	$\emptyset(t) = \frac{1}{d(t)}$
$b_1 + b_2 + b_3 + b_4 = 1$	f	1		f	$\sum_{i=1}^{4} b_i = 1$
$b_2c_2 + b_3c_3 + b_4c_4 = 1/2$	ff_y	2		$f'f$	$\sum_{i=2}^{4} b_i\, c_i = 1/2$
$b_3c_2a_{32} + b_4c_2a_{42} + b_4c_3a_{43} = 1/6$	ff_y^2	3		$f'f'f$	$\sum_{i=3,j=2}^{4,3} b_i\, a_{ij}c_j = 1/6$
$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = 1/3$	f^2f_{yy}	3		$f''(f,f)$	$\sum_{i=2}^{4} b_i\, c_i^2 = 1/3$
$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = 1/4$	f^3f_{yyy}	4		$f'''(f,f,f)$	$\sum_{i=2}^{4} b_i\, c_i^3 = 1/4$
$b_4a_{43}a_{32}c_2 = 1/24$	ff_y^3	4		$f'f'f'f$	$\sum_{i=4,j=3,k=2}^{4,3,2} b_i\, a_{ij}a_{jk}c_k = 1/24$
$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 = 1/12$	$f^2f_yf_{yy}$	4		$f'f''(f,f)$	$\sum_{i=3,j=2}^{4,3} b_i\, a_{ij}c_j^2 = 1/12$
$b_3a_{32}c_2c_3 + b_4a_{42}c_2c_4 + b_4a_{43}c_3c_4 = 1/8$	$f^2f_{yy}f_y$	4		$f''(f',f,f)$	$\sum_{i=3,j=2}^{4,3} b_i\, c_ia_{ij}c_j = 1/8$

The table above comprises of the eight equations generated using rooted trees, hence, they are resolved below:

$b_1 = 1/6$, $b_2 = 2/6$, $b_3 = 2/6$, $b_4 = 1/6$

$a_{21} = \frac{1}{2}$, $a_{31} = -\frac{1}{2}$, $a_{32} = 1$, $a_{41} = 1$, $a_{42} = -\frac{1}{2}$, $a_{43} = \frac{1}{2}$.

Hence, the fourth-stage fourth-order explicit Runge Kutta method ($f(y)$ functional derivatives) becomes:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_1 = f(y_n)$$
$$k_2 = f(y_n + \frac{h}{2}k_1)$$
$$k_3 = f(y_n + \frac{h}{2}(-k_1 + 2k_2))$$
$$k_4 = f(y_n + \frac{h}{2}(2k_1 - k_2 + k_3))$$

FINDINGS/CONTRIBUTION TO KNOWLEDGE

This research reveals that instead of the stress in expanding the f(y) functional derivatives and f(x,y) functional derivatives together, a researcher can decide to expand only the f(y) functional derivatives or f(x,y) functional derivatives independently and get the same set of equations as against the stress of expanding both together.

This is an eye opener, thus, making the derivation of any explicit Runge Kutta formula easier and more precise.

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