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PARAMETRIC VARIATION OF AN EXPLICIT FOURTH-STAGE FOURTH-ORDER RUNGE-KUTTA METHOD WITH ABSOLUTE STABILITY

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ABTRACT

The purpose of this paper is to separate the $f(y)$ functional derivatives by discarding all functional derivatives of $f(x, y)$ after using Taylor Series expansion to expand the fourth-stage fourth-order explicit Runge-Kutta method so as to derive a reduced number of equations for easy computation. Efforts will be made to vary the parameters with the aim of getting a new explicit fourth-order formula that can improve results when implemented on initial-value problems. Efforts will also be made to carry out stability, convergence and consistency analysis and represent the derived equations, their individual $f(y)$ functional derivatives and their various elementary differentials on Butcher's rooted trees. This idea is derivable from general graphs and combinatorics.

Keywords: *Rooted tree diagram, Comparison, Variation, explicit, y partial derivatives, Runge-Kutta Methods, Linear and non- linear equations, Taylor series, Graphs, Parameters, Initial-value Problems, Combinatorics, $f(y)$ functional derivatives, elementary differentials.*

INTRODUCTION

This research centers on separating the rooted trees and equations generated from the y functional derivatives by discarding all functional derivatives of $f(x,y)$, so as to enable us see if we can adopt a simpler approach in deriving a new explicit fourth-order Runge-Kutta formula that can improve results. It also has to do with the variation of the parameters in the derived equations generated after using Taylor's series expansion to expand the order. Scientific implementation of the formula on initial-value problems of the form: $y^1 = f(x, y), \; y(x_0) = y_0, \; a \leq x \leq b$, is also considered with a view to finding its stability, consistency and convergence. The essence is to see if the formula can improve results.

Recent works on Runge-Kutta analysis include Agbeboh (2006,2010), Agbeboh, Ukpebor and Esekhaigbe (2009), Esekhaigbe (2007) and Butcher(2003). More recent works are that of Van Der Houwen and Sommeijer (2013, 2014, 2015). The work of Butcher (1963,1966, 1987, 2009, 2010) revealed much successes in the analysis of explicit Runge-Kutta methods and their transformation to rooted tree diagrams. This was because the continuation of the process of Taylor Series gives rise to very complicated formulae. It was therefore, advantageous to use a graphical representation for a convenient analysis of the order of a Runge – kutta method; hence, the basic tree theory was introduced. A tree is a rooted graph which contains no circuits. The symbol T is used to represent the tree with only one vertex. All rooted trees can be represented using T and the operation [$t_1, t_2, \ldots, t_m$]. Hence, it is the differentials and equations derived that are represented on trees so as to enable us compare the order condition with their differentials for varying parameters.

Conclusively, despite the fact that good, reliable explicit Runge-Kutta formulas exist, there is still need for their transformation to rooted tree diagrams.

METHOD OF DERIVATION

i. From the general Runge-Kutta method, get a Fourth Stage-Fourth order method
ii. Obtain the Taylor series expansion of $k_{i's}^{:}$ about the point (x_n, y_n), i=2,3,4,
iii. Carry out substitution to ensure that all the $k_{i's}^{:}$ are in terms of k_1 only.
iv. Insert the $k_{i's}^{:}$ in terms of k_1 only into $b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$
v. Separate all f(y) functional derivatives with their coefficients from all f(x,y) functional derivatives and their coefficients.
vi. Discard all $f(x,y)$ functional derivatives and their coefficients.
vii. Equate the coefficients of all $f(y)$ functional derivatives with the coefficients Taylor series expansion involving only $f(y)$ functional derivatives of the form:
viii. $\emptyset_T(x,y,h) = f + \frac{h}{2!}ff_y + \frac{h^2}{3!}\left(ff_y^2 + f^2f_{yy}\right) + \frac{h^3}{4!}(4f^2 f_y f_{yy} + ff_y^3 + f^3f_{yyy}) + \frac{h^4}{5!}(7f^3f_yf_{yyy} + 4f^3f_{yy}^2 + 11f^2f_y^2f_{yy} + ff_y^4 + f^4f_{yyyy})$ (1)

As a result, a set of linear/non-linear equations will be generated. Represent those equations and their f(y) functional derivatives on Butcher's rooted tree.

ix. Vary the set of equations to derive a new fourth-stage fourth-order explicit Runge-Kutta formula.

DERIVATION OF THE FOURTH–ORDER FOURTH-STAGE ERK METHOD

According to Lambert (1991), the general R–Stage Runge–Kutta method is:

$$y_{n+1} = y_n + h\,\phi\,(x_n, y_n, h),$$

$$\phi\,(x_n, y_n, h) = \sum_{r=1}^{R} b_r\, k_r,$$

$$k_1 = f(x,y)$$

$$k_r = f(x + hc_r, y + h\sum_{s=1}^{r-1} a_{rs}k_s), r = 2, 3, \ldots R$$

The formula is defined by the number of stages s, the nodes$[c_r]^s{}_{r=1}$, the internal weights $[a_{rs}]^{r-1,s}{}_{s=1,r=2}$ and the external weights$[b_r]^s{}_{r=1}$.

From the above scheme, the fourth stage fourth – order method is:

$$y_{n+1} = y_n + h\,(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4)$$

$$k_1 = f\,(x_n, y_n)$$

$$k_2 = f\,(x_n + c_2, y_n + ha_{21}k_1)$$

$k_3 = f\,(x_n + c_3h, y_n + h(a_{31}k_1 + a_{32}k_2))$

$k_4 = f\,(x_n + c_4h, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3))$

Using Taylor's series expansion for $k_i's$, we have:

$k_1 = f\,(x_n, y_n)$

$k_2 = \sum_{r=o}^{\infty} \frac{1}{r!}(c_2h\,\frac{d}{dx} + ha_{21}k_1\frac{d}{dy})^r f(x_n, y_n)$

$k_3 = \sum_{r=o}^{\infty} \frac{1}{r!}(c_3h\,\frac{d}{dx} + h(a_{31}k_1 + a_{32}k_2)\frac{d}{dy})^r f(x_n, y_n)$

$k_4 = \sum_{r=o}^{\infty} \frac{1}{r!}(c_4h\,\frac{d}{dx} + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\frac{d}{dy})^r f(x_n, y_n)$

Hence, we have:

$k_1 = f$

$k_2 = f + \left(c_2hf_x + ha_{21}k_1f_y\right) + \frac{1}{2!}\left(c_2hf_x + ha_{21}k_1f_y\right)^2 + \frac{1}{3!}\left(c_2hf_x + ha_{21}k_1f_y\right)^3 + \frac{1}{4!}\left(c_2hf_x + ha_{21}k_1f_y\right)^4 + 0(h^5)$

$k_3 = f + \left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right) + \frac{1}{2!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^2 + \frac{1}{3!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^3 + \frac{1}{4!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^4 + 0(h^5)$

$k_4 = f + \left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right) + \frac{1}{2!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^2 + \frac{1}{3!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^3 + \frac{1}{4!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^4 + 0(h^5)$

Expanding fully and substituting the various k_i' s, i = 2, 3, 4 into their various positions in terms of k_1 only and collecting like terms, in terms of y derivatives only and discarding all $f(x, y)$ functional derivatives, we have:

$k_1 = f,\qquad k_2 = f + ha_{21}ff_y + \frac{h^2}{2!}a_{21}^2f^2f_{yy} + \frac{h^3}{3!}a_{21}^3f^3f_{yyy} + \frac{h^4}{4!}a_{21}^4f^4f_{yyyy},$

$k_3 = f + h\,(a_{31} + a_{32})ffy + h^2a_{21}a_{32}ff_y^2 + \frac{h^2}{2!}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{yy} + \frac{h^3}{3!}a_{21}a_{32}\big(a_{21} + 2(a_{31} + a_{32})\big)f^2f_yf_{yy} + \frac{h^3}{3!}(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3f_{yyy} + \frac{h^4}{3!}(a_{32}a_{21}^3 + 3a_{31}^2a_{32}a_{21} + 3a_{32}^3a_{21} + 6a_{31}a_{32}^2a_{21})f^3f_yf_{yyy} + \frac{h^4}{2!}a_{21}^2a_{32}(a_{31} + a_{32})f^3f_{yy}^2 + \frac{h^4}{2!}a_{32}^2a_{21}^2f^2f_y^2f_{yy} + \frac{h^4}{4!}(a_{31}^4 + 4a_{31}^3a_{32} + 6a_{31}^2a_{32}^2 + 4a_{31}a_{32}^3 + a_{32}^4)f^4f_{yyyy},$

$k_4 = f + h\,(a_{41} + a_{42} + a_{43})ff_y + h^2(a_{21}a_{42} + a_{31}a_{43} + a_{32}a_{43})ff_y^2 + \frac{h^2}{2!}(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + 2a_{42}a_{43} + a_{42}^2 + a_{43}^2)f^2f_{yy} + \frac{h^3}{2!}(a_{21}^2a_{42} + a_{31}^2a_{43} + 2a_{31}a_{32}\,a_{43} + a_{32}^2a_{43} + 2a_{21}a_{41}\,a_{42} + 2a_{31}a_{41}\,a_{43} + 2a_{32}a_{41}\,a_{43} + 2a_{31}a_{42}\,a_{43} + 2a_{32}a_{42}\,a_{43} + 2a_{21}a_{42}\,a_{43} + 2a_{21}a_{42}^2 + 2a_{31}a_{43}^2 + 2a_{32}a_{43}^2)f^2f_yf_{yy} + h^3a_{21}a_{32}\,a_{43}ff_y^3 + \frac{h^3}{2!}(a_{41}^3 + 3a_{41}^2a_{42} + 3a_{41}^2a_{43} + 3a_{41}a_{42}^2 + 6a_{41}a_{42}a_{43} + 3a_{42}^2a_{43} + 3a_{41}a_{43}^2 + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3)f^3f_{yyy} + \frac{h^4}{3!}\,(a_{31}^3a_{43} + 3a_{31}^2a_{32}a_{43} + 3a_{31}a_{32}^2a_{43} + 3a_{21}a_{41}^2a_{42} + 3a_{31}a_{41}^2a_{43} + 3a_{32}a_{41}^2a_{43} + 6a_{21}a_{41}a_{42}^2 + 6a_{31}a_{41}a_{42}a_{43} + 6a_{32}a_{41}a_{42}a_{43} + 6a_{21}a_{41}a_{42}a_{43} + 3a_{31}a_{41}^2a_{43} + a_{42}a_{21}^3 + 3a_{32}a_{42}^2a_{43} + 6a_{21}a_{42}^2a_{43} + 6a_{31}a_{41}a_{43}^2 + 6a_{32}a_{41}a_{43}^2 + 6a_{31}a_{42}a_{43}^2 + 6a_{32}a_{42}a_{43}^2 + 3a_{21}a_{42}a_{43}^2 + 3a_{21}a_{43}^3 + 3a_{31}a_{43}^3 + 3a_{32}a_{43}^3)f^3f_yf_{yyy} + \frac{h^4}{2!}(a_{21}^2a_{32}a_{43} + 2a_{21}a_{31}a_{32}a_{43} + 2a_{21}a_{32}^2a_{43} + 2a_{21}a_{32}a_{41}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + 2a_{21}a_{31}a_{42}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + a_{21}^2a_{42}^2 + a_{21}a_{32}a_{43}^2 + a_{31}^2a_{43}^2 + 2a_{31}a_{32}a_{43}^2 + a_{32}^2a_{43}^2)f^2f_y^2f_{yy} + \frac{h^4}{2!}\Big(a_{21}^2a_{41}a_{42} + a_{31}^2a_{41}a_{42} + 2a_{31}a_{32}a_{41}a_{43} + a_{32}^2a_{41}a_{43} + a_{31}^2a_{41}a_{43} + 2a_{31}a_{32}a_{42}a_{43} + a_{32}^2a_{42}a_{43} + \frac{a_{21}^2a_{42}^2}{2!} + \frac{a_{31}^2a_{43}^2}{2!} + a_{31}a_{32}a_{43}^2 + \frac{a_{31}^2a_{43}^2}{2!}\Big)f^3f_{yy}^2 + \frac{h^4}{4!}\,(a_{41}^4 + 4_{41}^3a_{42} + 4a_{41}^3a_{43} + 6a_{41}^2a_{41}^2 + 12a_{41}^2a_{42}a_{43} + 6a_{42}^2a_{43}^2 + 4a_{41}a_{43}^3 + 4a_{42}a_{43}^3 + 12a_{41}a_{42}^2a_{43} + 2a_{41}a_{42}a_{43}^2 + 6a_{41}^2a_{43}^2 + 4a_{42}^3a_{43} + 4a_{41}a_{42}^3 + a_{42}^4 + a_{43}^4)\,f^4f_{yyyy}$

Putting the $k_{i's}^{;}$ *(y derivatives only) into* $y_{n+1} = y_n + h(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$ *where* $\phi(x, y, h) = b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$ *and equating coefficients with the Taylor series expansion:*

$\emptyset_T(x, y, h) = f + \frac{h}{2!} f f_y + \frac{h^2}{3!}(f f_y^2 + f^2 f_{yy}) + \frac{h^3}{4!}(4 f_y^2 f_y f_{yy} + f f_y^3 + f^3 f_{yyy}) + \frac{h^4}{5!}(4 f^3 f_y f_{yyy} + 4 f^3 f_{yy}^2 + 11 f^2 f^2 f_{yy} + f f_y^4 + f^4 f_{yyyy})$, we have the following equations:

$b_1 + b_2 + b_3 + b_4 = 1$ (1)

$b_2 a_{21} + b_3(a_{31} + a_{32}) + b_4(a_{41} + a_{42} + a_{43}) = ½$ (2)

$b_3 a_{21} + b_4(a_{21} a_{41} + a_{43}(a_{31} + a_{32}) = 1/6$ (3)

$b_2 a_{21}^2 + b_3(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2) + b_4(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + 2a_{42}a_{43} + a_{42}^2 + a_{43}^2) = \frac{1}{3}$ (4)

$b_2 a_{21}^3 + b_3(a_{31}^3 + 3a_{31}^2 a_{32} + 3a_{31}a_{32}^2 + a_{32}^3) + b_4(a_{41}^3 + 3a_{41}^2 a_{42} + 3a_{42}^2 a_{43} + 3a_{41}a_{42}^2 + 6a_{41}a_{42}a_{43} + 3a_{42}^2 a_{43} + 3a_{41}a_{43}^2 + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3) = ¼$. (5)

$b_3 a_{21} a_{32}\big(a_{21} + 2(a_{31} + a_{32})\big) + b_4(a_{41}^2 a_{42} + a_{43}(a_{31} + a_{32})^2 + 2a_{21}a_{42}(a_{41} + a_{42} + a_{43}) + 2a_{31}4_3(a_{41} + a_{42} + a_{43}) + 2a_{32}a_{43}(a_{41} + a_{42} + a_{43})) = 1/3$ (6)

$b_4 a_{21} a_{32} a_{43} = 1/24$ (7)

Now from (1), setting $b_1 = b_4 = 1/6$, $b_2 = b_3 = 2/6$

(8) becomes: $2A + 2B + P = 3$ (14)

(10) becomes: $2A^2 + 2B^2 + P^2 = 2$ (15)

(11) becomes: $2A^3 + 2B^3 + P^3 = 3/2$ (16)

From (14), (15), (16)

$A = ½$, $B = ½$, $P = 1$,

Hence (9) becomes: $2a_{32} + a_{42} + a_{43} = 2$ (17)

(13) becomes: $a_{32} a_{43} = ½$ (18)

(12) becomes: $6a_{32} + 5a_{42} + a_{43} + 8a_{43}a_{43} + 8a_{42}a_{43} = 8$ (19)

From (18), let $a_{43} = \frac{1}{2}$, then $a_{32} = 1$, From (17), $a_{42} = -\frac{1}{2}$, But $A = ½$, $B = ½$, $P = 1$

Therefore, $a_{21} = \frac{1}{2}$, $a_{31} + a_{32} = \frac{1}{2}$, $\rightarrow a_{31} = -\frac{1}{2}$, $a_{41} + a_{42} + a_{43} = 1$, $a_{41} = 1$

In conclusion, $b_1 = 1/6$, $b_2 = 2/6$, $b_3 = 2/6$, $b_4 = 1/6$

$$a_{21} = \frac{1}{2}, \quad a_{31} = -\frac{1}{2}, \quad a_{32} = 1, \quad a_{41} = 1, \quad a_{42} = -\frac{1}{2}, \quad a_{43} = \frac{1}{2}.$$

THE FOURTH- ORDER (Y DERIVATIVES ONLY) BECOMES:

$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = f(y_n)$

$k_2 = f(y_n + \frac{h}{2} k_1)$

$k_3 = f(y_n + \frac{h}{2}(-k_1 + 2k_2))$

$k_4 = f(y_n + \frac{h}{2}(2k_1 - k_2 + k_3))$

0				
0	½			
0	- ½	1		
0	1	-½	½	
	1/6	1/3	1/3	1/6

3.1 proof for stability of the method:

From the Formula above,

$$k_1 = \lambda y,\ k_2 = f\left(y_n + \frac{h}{2}k_1\right) = \lambda(y_n + \frac{h\lambda y}{2})$$

$$k_2 = \lambda y\left(1 + \frac{\lambda h}{2}\right)$$

$$k_3 = f\left(y_n + \frac{h}{2}(-k_1 + 2k_2)\right) = \lambda\left(y_n - \frac{h\lambda y}{2} + \frac{2h}{2}\left(\lambda y\left(1 + \frac{\lambda h}{2}\right)\right)\right)$$

$$k_3 = \lambda\left(y_n - \frac{\lambda y h}{2} + h\lambda y + \frac{h^2\lambda^2 y}{2}\right)$$

$$k_3 = \lambda y\left(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2 y}{2}\right)$$

$$k_4 = f\left(y_n + \frac{h}{2}(2k_1 - k_2 + k_3)\right) + \lambda(y_n + h\lambda y - \frac{\lambda y h}{2}\left(1 + \frac{h\lambda}{2}\right) + \frac{\lambda y h}{2}(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{2}))$$

$$k_4 = \lambda\left(y_n + h\lambda y - \frac{\lambda y h}{2} - \frac{h^2\lambda^2 y}{4} + \frac{\lambda y h}{2} + \frac{h^2\lambda^2 y}{4} + \frac{h^3\lambda^3 y}{4}\right)$$

$$k_4 = \lambda\left(y_n + h\lambda y + \frac{h^3\lambda^3 y}{4}\right)$$

$$k_4 = \lambda y\left(1 + \lambda h + \frac{h^3\lambda^3 y}{4}\right)$$

$$then,\ y_{n+1} - y_n = \frac{h}{6}\left[\lambda y + 2\lambda y\left(1 + \frac{h\lambda}{2}\right) + 2\lambda y\left(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{4}\right) + \lambda y\left(1 + \lambda y + \frac{h^3\lambda^3}{4}\right)\right]$$

$$y_{n+1} - y_n = \frac{\lambda y h}{6}\left[1 + 2\left(1 + \frac{h\lambda}{2}\right) + 2\left(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{2}\right) + 1\left(1 + \lambda y + \frac{h^3\lambda^3}{4}\right)\right]$$

$$y_{n+1} - y_n = \frac{\lambda y h}{6}\left[6 + 3\lambda h + h^2\lambda^2 + \frac{h^3\lambda^3}{4}\right]$$

Dividing by y and setting $\mu = \lambda h,\ we\ have$:

$$\frac{y_{n+1} - y_n}{y_n} = \frac{\mu}{6}\left[6 + 3\lambda h + h^2\lambda^2 + \frac{h^3\lambda^3}{4}\right]$$

$$\frac{y_{n+1}}{y_n} - 1 = \left[\mu + \frac{\mu^2}{2} + \frac{\mu^3}{6} + \frac{\mu^4}{24}\right]$$

$$\frac{y_{n+1}}{y_n} = 1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \frac{\mu^4}{4!} = 0$$

Hence, we have the stability polynomial, which is the same as the Classical fourth-order method.

Resolving the above polynomial using MAPLE, we have the complex roots as follows:

-1.729444231067705456600-0.88897437612186582717i,

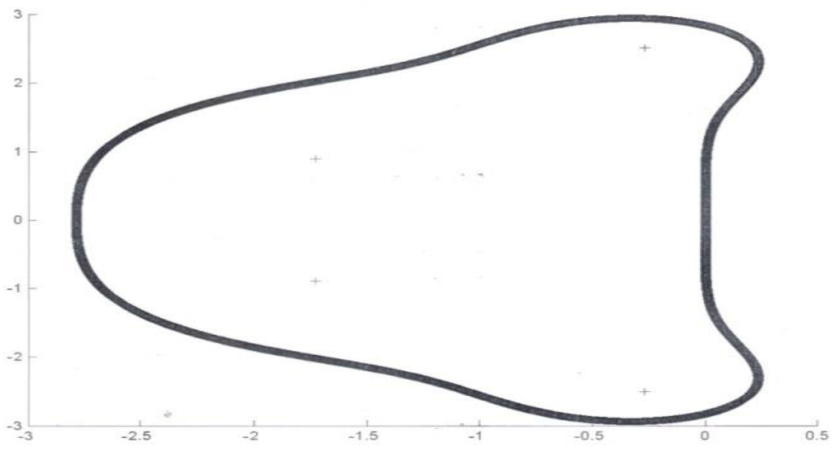
-1.72944423106770545660+0.88897437612186582717i,

-0.27055576893229454343-2.50477590436243448970i,

-0.27055576893229454343+2.50477590436243448970i.

Plotting the complex roots on a graph (the real parts on the x-axis and the imaginary parts on the y-axis) using MATLAB CODE, we have the absolute stability region seen in the diagram below:

Figure 1: Region of Absolute Stability For the fourth-Stage fourth-order ((y) derivatives only)



4.0 Implementation of the Formula and Results

The formula is implemented on the initial – value problems below with the aid of FORTRAN programming language:

(i) $y^1 = -y, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = \frac{1}{e^{xn}}$

(ii) $y^1 = y, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = e^{xn}$

(iii) $y^1 = 1 + y^2, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = \tan\left(x_n + {}^{\pi}/_{4}\right), h = 0.1$

TABLE 2 TABLES OF RESULTS

PROBLEM 1

XN	YN	TSOL	ERROR
.1D+00	0.9048375000000D+00	0.9048374180360D+00	-.8196404044369D-07
.2D+00	0.8187309014063D+00	0.8187307530780D+00	-.1483282683346D-06
.3D+00	0.7408184220012D+00	0.7408182206817D+00	-.2013194597694D-06
.4D+00	0.6703202889175D+00	0.6703200460356D+00	-.2428818514089D-06
.5D+00	0.6065309344234D+00	0.6065306597126D+00	-.2747107467060D-06
.6D+00	0.5488119343763D+00	0.5488116360940D+00	-.2982822888686D-06
.7D+00	0.4965856186712D+00	0.4965853037914D+00	-.3148798197183D-06
.8D+00	0.4493292897344D+00	0.4493289641172D+00	-.3256172068089D-06
.9D+00	0.4065699912001D+00	0.4065696597406D+00	-.3314594766990D-06
.1D+01	0.3678797744125D+00	0.3678794411714D+00	-.3332410563051D-06

PROBLEM 2

XN	YN	TSOL	ERROR
.1D+00	0.1105170833333D+01	0.1105170918076D+01	0.8474231405486D-07
.2D+00	0.1221402570851D+01	0.1221402758160D+01	0.1873094752636D-06
.3D+00	0.1349858497063D+01	0.1349858807576D+01	0.3105134649406D-06
.4D+00	0.1491824240081D+01	0.1491824697641D+01	0.4575605843105D-06
.5D+00	0.1648720638597D+01	0.1648721270700D+01	0.6321032899326D-06

.6D+00	0.1822117962092D+01	0.1822118800391D+01	0.8382985758892D-06
.7D+00	0.2013751626597D+01	0.2013752707470D+01	0.1080873699877D-05
.8D+00	0.2225539563292D+01	0.2225540928492D+01	0.1365200152481D-05
.9D+00	0.2459601413780D+01	0.2459603111157D+01	0.1697376878607D-05
.1D+01	0.2718279744135D+01	0.2718281828459D+01	0.2084323879270D-05

PROBLEM 3

XN	YN	TSOL	ERROR
.1D+00	0.1223051005569D+01	0.1223048880450D+01	-.2125119075158D-05
.2D+00	0.1508502732390D+01	0.1508497647121D+01	-.5085268468541D-05
.3D+00	0.1895771003842D+01	0.1895765122854D+01	-.5880987590245D-05
.4D+00	0.2464942965339D+01	0.2464962756723D+01	0.1979138375674D-04
.5D+00	0.3407951033347D+01	0.3408223442336D+01	0.2724089890727D-03
.6D+00	0.5328707710968D+01	0.5331855223459D+01	0.3147512490389D-02
.7D+00	0.1159500710295D+02	0.1168137380031D+02	0.8636669735614D-01
.8D+00	0.2841447010395D+03	-.6847966834558D+02	-.3526243693850D+03
.9D+00	0.8635045424394D+20	-.8687629546482D+01	-.8635045424394D+20
.1D+01	0.1640237043432+300	-.4588037824984D+01	-.1640237043432+300

FINDINGS AND CONTRIBUTION TO KNOWLEDGE

This study reveals that expanding f(y) functional derivatives can generate a formula that can improve the performance of results. After our implementation, it shows from the tables of numerical results that the method is highly efficient. The proof for stability also shows that the method is absolutely stable. The stability polynomial and stability curve also show that our method compares favourably well with the well known classical fourth-order Runge-Kutta method, and as such, generating more accurate results when implemented on initial-value problems in ordinary differential equations.

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