DOI: https://doi.org/10.53555/nnms.v8i9.1033 **Publication URL: http://nnpub.org/index.php/MS/article/view/1033**

GENERALISED SEMI CLOSED SETS IN GRILL TOPOLOGICAL SPACES

S. Vibin Salim Raj[1]*, V. Senthil kumaran[2], Y. Palaniappan[3]

*[1] *M. Phil, Scholar, Arignar Anna Government Arts College, Musiri, Tamilnadu, India Email: rajvibin10@gmail.com*

[2]*Associate Professor of Mathematics, Arignar Anna Government Arts College, Musiri, Tamilnadu, India. Email: vsenthil1966@gmail.com*

[3]*Associate Professor of Mathematics (Retd.), Arignar Anna Government Arts College, Musiri, Tamilnadu, India Email: palaniappany48@gmail.com*

Corresponding Author: -
Email: rajvibin10@gmail.com

Abstract: -

The purpose of this paper is to introduce and investigate a new class of generalized semi closed sets in terms of Grill G on X. The characterization of such sets along with certain other properties of them are obtained.

Keywords: *gs closed, topology* τ_G, *operator* ϕ, $G_{(gs)^*}$ *closed. 2010 AMS subject classification: 54B05, 54C05*

1. INTRODUCTION

It is found from literature that during recent years many topologists are interested in the study of generalized types of closed sets. For instance, a certain form of generalized closed sets was initiated by Levine [6] whereas the notion of generalized *semi closed (g*s closed) set was studied by Veerakumar [9]. Following the trend, we have introduced and investigated a kind of generalized closed sets, the definition being formulated in terms of grills. The concept of grill was introduced by Choquet [1] in the year 1947. From subsequent investigations it is revealed that grills can be used as an extremely useful device for investigation of a number of topological problems.

2. PRELIMINARIES

Definition 2.1: A nonempty collection G of non-empty subsets of a topological space X is called a grill [1] if
(i) $A \in G$ and $A \subseteq B \subseteq X \Rightarrow B \in G$ and
(ii) $A, B \subseteq X$ and $A \cup B \in G \Rightarrow A \in G$ or $B \in G$

Let G be a grill on a topological space (X, τ). In [7] an operator $\phi: P(X) \to P(X)$ was defined by $\phi(A) = \{x \in X / U \cap A \in G, \forall U \in \tau(x)\}$, $\tau(x)$ denotes the neighborhood of x. Also the map $\psi : P(x) \to P(x)$, given by $\psi(A) = A \cup \phi(A)$ for all $A \in P(X)$.
Corresponding to a grill G, on a topological space (X, τ) there exists a unique topology τ_G on X given by $\tau_G = \{U \subseteq X / \psi(X \setminus U) = X \setminus U\}$ where for any $A \subseteq X, \psi(A) = A \cup \phi(A) = \tau_G - cl(A)$.
Thus a subset A of X is τ_G -closed (resp. τ_G -dense in itself) if ψ (A)= A or equivalently if $\phi(A) \subseteq A$ (resp. $A \subseteq \phi(A)$)

In the next section, we introduce and analyse a new class of generalized closed sets, namely G(gs)* closed sets, in terms of a given grill G, the definition having a close bearing to the above operator ϕ .
We introduce and investigate the notion of (gs)* continuous functions in grill topological spaces. Also, we investigate the relationship with other functions.
Throughout the paper, by a space X we always mean a topological space (X,τ) with no separation properties assumed. If $A \subseteq X$, we shall adopt the usual notations int(A) and cl(A) respectively for the interior and closure of A in (X,τ). Again τ_G – Cl(A) and τ_G -int(A) will respectively denote the closure and interior of A in (X, τ_G). Similarly, whenever we say that a subset A of a space X is open (or closed) it will mean that A is open (or closed) in (X, τ). For open and closed sets with respect to any other topology on X, eg. τ_G, we shall write τ_G - open and τ_G -closed. The collection of all open neighborhoods of a point x in (X, τ) will be denoted by $\tau(x)$.
(X, τ, G) denotes a topological space (X, τ) with a grill G.

Definition 2.2: Let (X,τ) be a topological space. A subset A of X is said to be
(1) semiclosed if int cl(A) $\subseteq$ A
(2) generalized closed (g closed) of cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is open in X.
(3) generalized semi closed (gs closed) of scl A $\subseteq$ U whenever $A \subseteq U$ and U is open in X.
(4) θ -closed if A = θ cl A where

$$\theta \text{ cl } A = \{x \in X : cl(U) \cap A \neq \phi , \forall U \in \tau \text{ and } x \in U\}$$

(5) δ -closed if A = δ cl A where

$$\delta \text{ cl } A = \{x \in X; \text{int cl } (U) \cap A \neq \phi , \forall U \in \tau \text{ and } x \in U\}$$

The complements of the above closed sets are respective open sets.

Definition 2.3: A function $f : (X, \tau) \to (Y, \sigma)$ is said to be
(1) gs continuous if $f^{-1}(U)$ is gs open in X, for every open set U of Y.
(2) θ continuous if $f^{-1}(U)$ is θ open in X, for every open set U of Y.
(3) δ -continuous if $f^{-1}(U)$ is δ -open in X for every open set U of Y.

Definition 2.4: A function $f : (X,\tau) \to (Y, \sigma)$ is said to be
(1) (gs)* closed if f (F) is (gs)* closed in Y, for every closed set F of X,
(2) τ_G closed if f (F) is τ_G closed in Y, for every closed set F of X.
(3) θ closed if f (F) is θ closed in Y, for every closed set F of X.
(4) δ closed if f (F) is δ closed in Y, for every closed set F of X.

Theorem 2.5: [7] Let (X,τ) be a topological space and G be a grill on X. Then for any $A, B \subseteq X$ following hold
(a) $A \subseteq B \Rightarrow \phi(A) \subseteq \phi(B)$
(b) $\phi(A \cup B) = \phi(A) \cup \phi(B)$
(c) $\phi(\phi(A) \subseteq \phi(A) = cl(\phi(A)) \subseteq cl(A)$

3. G(gs)* CLOSED SETS

Definition 3.1: A subset A of a topological space (X,τ) is said to be (gs)* closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gs open in X.

Definition 3.2: Let (X,τ) be a topological space and G be a grill on X. Then the subset A of (X,τ) is said to be (gs)* closed with respect to a grill G(G(gs)* closed) if $\phi(A) \subseteq U$ whenever $A \subseteq U$ and U gs open in X.
The complement of G(gs)* closed set in X is called G(gs)* open in X.

Theorem 3.3: Let (X,τ) be a topological space and G be a grill on X. Then
(1) Every closed set in X in G(gs)* closed
(2) For A, G(gs)* closed in X, ϕ (A) is G(gs)* closed.
(3) Every τ_G closed set isG(gs)* closed.
(4) Any non-member of G is G(gs)* closed.
(5) Any (gs)* closed set is G(gs)* closed.
(6) Every θ closed set is G(gs)* closed.
(7) Every δ closed set is G(gs)* closed

Proof:
(1) Let A be closed. Then cl (A) = A. Let $A \subseteq U$ where U is gs open.
$\phi(A) \subseteq cl\, A = A \subseteq U$. A is $G_{(gs)}$* closed.
(2) A is G(gs)* closed closed. $\phi(A) \subseteq U$, U is gs open. $\phi(\phi(A)) \subseteq \phi(A) \subseteq U$. Hence ϕ (A) is G(gs)*
(3) Let A be τ_G closed. Let $A \subseteq U$, where U is gs open. As A is τ_G closed, $\phi(A) \subseteq A \subseteq U$. Hence A is G(gs)* closed.
(4) Let $A \notin G$. Let $A \subseteq U$, U is gs open. $\phi(A) = \phi \subseteq U$. Hence A is G(gs)* closed.
(5) Let $A \subseteq U$, U is gs open. A is (gs)* closed. Hence $cl\, A \subseteq U$. So $\phi(A) \subseteq cl\, A \subseteq U$. Therefore A is G(gs)* closed.
(6) Let $A \subseteq U$, U is gs open. A is θ closed $\phi(A) \subseteq cl\, A \subseteq \theta \subseteq cl\,(A) = A \subseteq U$. Hence A is G(gs)* closed.
(7) Let $A \subseteq U$, U is gs open A is δ closed $\phi(A) \subseteq cl\, A \subseteq \delta cl\, A = A \subseteq U$. Hence A is G(gs)* closed. The converse of the above statements need not be true can be seen from the following examples.

Example 3.4: Let X = {a, b, c}
τ = {ϕ, {a}, {a, b}, X}
G = {{a}, {a, b}, {a, c}, X}, A = {b}. Let $A \subseteq U$ where U is gs open.
$\phi(A) = \phi \subseteq U$. Hence A is G(gs)* closed but not A = {b} is closed.

Example 3.5: Let X = {a, b, c},
τ = {ϕ, {a}, X}, G = {{a}, {a, b}, {a, c}, X}

A = {a}, $\phi(A)$ = {a, b, c} $\not\subseteq$ {a}.

A is not $G_{(gs)}$* closed. $\phi(A)$ is $G_{(gs)}$* closed.

Example 3.6: Let X = {a, b, c},
τ = {ϕ, {a, b}, X}
G = {{a}, {b}, {c}, {a, b}, {a, c}, {b, c}, X}
A = {b, c} A is G(gs)* closed but not τ_G closed.

Example 3.7: Refer example 3.6
A is G(gs)* closed but not a nonmember.

Example 3.8: Refer example 3.5 Let G = {{b, c}, X}
A is G(gs)* closed but not (gs)* closed.

Example 3.9: Refer example 3.5
A is G(gs)* closed but not □closed.

Example 3.10: Refer example 3.5
A is G(gs)* closed but not δ closed.

Lemma 3.11 : Let (X,τ) be a space and G be a grill on X. If $A \subseteq X$ is τ_G – dense in itself, then
$\phi(A) = cl\, \phi(A) = \tau_G - cl(A) = cl(A)$

Theorem 3.12: Let (X,τ) be topological space and G be a grill on X. Then for $A \subseteq X$, A is G(gs)* closed iff $\tau_G - cl(A) \subseteq U$, $A \subseteq U$ and U is gs open.
Proof: Suppose A is G(gs)* closed. Then $\phi(A) \subseteq U \Rightarrow A \cup \phi(A) \subseteq U$. Therefore, $\tau_G - cl(A) \subseteq U$, $A \subseteq U$ and U is gs open.
Conversely $\tau_G - cl(A) \subseteq U$, $A \subseteq U$, U is gs open
Therefore $A \cup \phi(A) \subseteq U \Rightarrow \phi(A) \subseteq U$. Hence A is G(gs)* closed.

Theorem 3.13: Let G be a grill on a space (X,τ). If A is τ_G -dense in itself and G(gs)* closed implies A is (gs)* closed.
Proof: Let A be τ_G -dense in itself, then by lemma 3.1, $\phi(A) = cl(A)$.
Since A is G(gs)* closed, $\phi(A) \subseteq U$ where U is gs open in X and $A \subseteq U$.
Therefore $cl(A) \subseteq U$ where U is gs open in X and $A \subseteq U$. Hence A is (gs)* closed.

Theorem 3.14: For any grill G ona space (X,τ) the following are equivalent.
(a) Every subset of X is G(gs)* closed.
(b) Every gs open subset of (X,τ) is τ_G-closed.
Proof: (a) $\Rightarrow$ (b)
Let A be gs open in (X,τ). Then by (a), A is G(gs)* closed, so that $\phi(A) \subseteq A$. Therefore A is τ_G-closed.
(b) $\Rightarrow$(a). Let $A \subseteq X$ and U be gs open in (X,τ) such that $A \subseteq U$. Then (b), $\phi(U) \subseteq U$.
Also, $A \subseteq U \Rightarrow \phi(A) \subseteq \phi(U) \subseteq U$
There A is G(gs)* closed.

Theorem 3.15: Let (X,τ) be a topological space and G be a grill on X and A, B be subsets of X such that $A \subseteq B \subseteq \tau_G - cl(A)$. If A is G(gs)* closed then B is G(gs)* closed.
Proof: Suppose $B \subseteq U$ and U is gs open in X. Since A is G(gs)* closed.
$\phi(A) \subseteq U \Rightarrow \tau G - cl(A) \subseteq U$…. (1)
Now, $A \subseteq B \subseteq \tau G - cl(A)$ which implies $\tau_G - cl(A) \subseteq \tau_G - cl(B) \subseteq \tau_G - cl(A)$.
Therefore $\tau_G - cl(A) = \tau_G - cl(B)$
Therefore by (1) $\tau_G - cl(B) \subseteq U$. Hence B is G(gs)* closed.

Corollary 3.16: τ_G -closure of every G(gs)* closed set is G(gs)* closed.

Theorem 3.17: Let G be a grill on a space (X,τ) and A, B be subsets of X such that $A \subseteq B \subseteq \phi(A)$. If A is G(gs)* closed then A and B are gs closed.
Proof: Let $A \subseteq B \subseteq \phi(A)$, then $A \subseteq B \subseteq \tau G - cl(A)$. By theorem 3.15, B is G(gs)* closed. Again $A \subseteq B \subseteq \phi(A) \Rightarrow \phi(A) \subseteq \phi(B) \subseteq \phi(\phi(A)) \subseteq \phi(A)$. This implies that $\phi(A) = \phi(B)$. By theorem 3.13, A and B are gs closed.

Theorem 3.18: Let G be a Grill on a space (X,τ). Then a subset A of X is G(gs)* open iff $F \subseteq \tau_G - int(A)$ whenever $F \subseteq A$ and F is gs closed.
Proof: Let A be G(gs)* open set and $F \subseteq A$ where F is gs closed. Then $X \setminus A \subseteq X \setminus F$. This implies that $\phi(X \setminus A) \subseteq \phi(X \setminus F) = X \setminus F$. Hence $\tau_G - cl(X\setminus A) \subseteq X\setminus F$ which implies $F \subseteq \tau_G - int(A)$
Conversely, $F \subseteq \tau G - (int(A))$, $\tau G\ cl(X - A) \subseteq X - F$, $\phi(X - A) \subseteq X - F$, A is G(gs)* open.

Remark 3.19:
(a) Every continuous function is gs continuous.
(b) Every gs - continuous function is G(gs)* continuous.

Example 3.20: Refer example 3.5
Define f : $(X, \tau, G) \rightarrow (X, \tau)$ by f (a) = c, f (b) = a, f (c) = b f is gs continuous but not continuous as $f^{-1}(\{a\}) = \{b\}$ is not open. Define f by f (a) = c, f (b) = a, f (c) = a, f is G(gs)* continuous but not gs continuous as $f^{-1}(\{a\}) = \{b, c\}$ is not gs open.

Remark 3.21: Every □-continuous function is G(gs)* continuous.

Example 3.22: Refer example 3.5
Define *f* : $(X, \tau, G) \rightarrow (X, \tau)$ by *f* (a) = c, *f* (b) = a, *f* (c) = a. *f* is G(gs)* continuous but not □ continuous as $f^{-1}(\{a\}) = \{b, c\}$ is not □ open.

Remark 3.23: Every δ-continuous function is G(gs)* continuous.

Example 3.24: Refer example 3.22

f is G(gs)* continuous but not δ continuous as f^{-1} ({a}) = {b, c} is not δ open.

Definition 3.25: τ_G, (gs)* closed function.
A function *f* : (X,τ) □ (Y, τ, G) is said to be τ_G(gs)* closed if *f* (A) is τ_G (gs)* closed in Y for every closed set A in X.

Definition 3.26: A function *f* : (X,τ) □ (Y, τ, G) is said to be □□closed if *f* (A) is □ closed in Y for every closed set A in X.

Definition 3.27: A function *f* : (X,τ) □ (Y, τ, G) is said to be δ closed if *f* (A) is δ closed in Y for every closed set A in X.

Definition 3.28: A function *f* : (X,τ) □ (Y, τ, G) is said to be ϕ(G(gs)*) closed if *f* (A) is ϕ(G(gs)*) closed in Y for every closed set A in X.

Theorem 3.29:
1. Every closed function is G(gs)* closed function.
2. Every G(gs)* closed function is ϕ(G(gs)*) closed function.
3. Every τ_G closed function is G(gs)* closed function.
4. Every (gs)* closed function is G(gs)* closed function.
5. Every □ closed function is G(gs)* closed function.
6. Every δ closed function is G(gs)* closed function.

Proof: Obvious
The converse of the above statements need not be true can be seen from the following examples.

Example 3.30: Refer example 3.4
Define *f* : (X, τ) □ (X, τ, G) by f (a) = a, f (b) = c, f (c) = b. f is G(gs)*closed but not closed as *f* ({c}) = {b} is not closed in X.

Example 3.31: Refer example 3.4
Define *f* : (X, τ) □ (X, τ, G) by f (a) = a, f (b) = c, f (c) = b. f is ϕG(gs)* closed function but not G(gs)* closed function as*f* ({c}) = {a} is not G(gs)* closed.

Example 3.32: Refer example 3.6
Let f : (X, τ) □ (X, τ, G) be the identity function f is G(gs)* closed function but not τG closed function as f ({b, c}) = {b, c} is not τG closed.

Example 3.33: Refer example 3.6
Define *f* : (X,τ) □ (X, τ, G) by *f* (a) = b, *f* (b) = a, *f* (c) = a. *f* is G(gs)* closed function but not (gs)* closed function.

Example 3.34: Refer example 3.6
Define *f* : (X,τ) □ (X, τ, G) by *f* (a) = b, *f* (b) = a, *f* (c) = a. *f* is G(gs)* closed function but not (gs)* a closed function as *f* ({b, c}) = {a} is not (gs)* closed.

Example 3.35: Refer the previous example, *f* is G(gs)* closed function but not □ closed function as *f* ({b, c}) = {a} is not □ closed.

Example 3.36: Refer the previous example, *f* is G(gs)* closed function but not δ closed function as *f* ({b, c}) = {a} is not δ closed.

Theorem 3.37: If f : (X, τ) θ (Y, σ) is closed and g : (Y, σ) θ (Z, η, G) is G(gs)* closed then g o f : (X, τ) θ (Z, η, G) is G(gs)* is closed.

Theorem 3.38: A map *f* : X □ Y is G(gs)* closed if and only if for each subset S of Y and each open set U of X such that $f^{-1}(S) \subseteq U$, there is a G(gs)* open subset V of Y such that $S \subset V$ and $f^{-1}(V) \subseteq U$.
Proof: Let *f* be G(gs)* closed. Let $S \subseteq Y$ and U be an open set of X such that $f^{-1}(S) \subseteq U$. X – U is closed in X. *f* (X – U) is G(gs)* closed in Y. V = Y – *f* (X – U) is G(gs)* open in Y.
$f^{-1}(V) = X - f^{-1}(f(X - U)) \subseteq X - (X - U) = U$
Conversely, let F be closed in X. $f^{-1}(f(F^c)) \subseteq F^c$ and F^c is open in X.
By assumption, there exists a G(gs)* open subset V of Y such that $f(F^c) \subset V$ and $f^{-1}(v) \subseteq F^c$. This implies F

$\subseteq (f^{-1}(V))^c$

Hence $V^c \subseteq (f(F^c))^c = f(F) \subseteq f(f^{-1}(V))^c \subseteq V^c$

So, $f(F) = V^c$, which is G(gs)* closed.

Definition 3.39: Let X and Y be topological spaces. A map $f : X \to Y$ is called G(gs)* open map if the image of every open set of x is G(gs)* open in Y.

Theorem 3.40: For any bijection map $f : X \to Y$, the following are equivalent.

(1) $f^{-1} : Y \to X$ is G(gs)* continuous map

(2) f is G(gs)* open map

(3) f is G(gs)* closed map

Proof: (1) $\Rightarrow$ (2):

Let U be open in X $(f^{-1})^{-1}(U)$ is G(gs)* open in Y. That is $f(U)$ is G(gs)* open in Y.

(2) $\Rightarrow$ (3) :

Let F be a closed set of X. Then F^c is open in X.

By assumption

$f(F^c)$ is G(gs)* open in Y. $f(F^c) = (f(F))^c$ is G(gs)* open in Y.

$f(F)$ is G(gs)* closed in Y.

(3) $\Rightarrow$ (1):

Let F be closed in X. $f(F)$ is G(gs)* closed in Y. $f(F) = (f^{-1})^{-1}(F)$ is G(gs)* closed in Y. Hence f^{-1} is G(gs)* continuous map.

References:

[1]. G Choquet, “Sur les notions de filter et grille”, comptesRendus Acad. Sci. Paris, 224(1947), 171-173.
[2]. DhananjoyMandal and M.N. Mukherjee, “On a type of generalized closed sets”, Bol. Soc. Paran. Mat. 301 (2012), 67-76.
[3]. J. Dontchev and T.Noiri, “Quasi Normal spaces and □g-closed sets”, Acta. Math. Hunger, 89(3) (2000), 211-219.
[4]. J. Dontchev and H. Maki, “On □ -generalized closed sets”, Topology Atlass.
[5]. S. Fomin, “Extensions of topological spaces”, Ann. Math, 44 (1943), 471-480.
[6]. N. Levine, “Generalized closed sets in topology”, Rend. cire. Mat, Palermo, (2), 19(1970), 89–96.
[7]. B. Roy and M.N. Mukherjee, “On a typical topology induced by a grill”, Sooochow J. Math, 33(4) (2007), 771-786.
[8]. M.H. Stone, “Application of the theory of Boolean rings to general topology”, Trans, Amer, Math. Soc., 41 (1937), 374-481.
[9]. M.K.R.S. Veerakumar, “Between closed and *g* closed sets,” Mem. Fac. Sci Kochi. Univer. Math vol 21(2000), 1-19.