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A RELATIONSHIP BETWEEN THE CURVATURE TENSOR AND THE DIFFERENCE TENSOR FOR AFFINE HYPERSURFACES

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Abstract: -

In the present work, it is obtained a class of hypersurfaces, of decomposable type, for which the curvature tensor associated to the affine normal connection, and the Levi-Civita covariant derivative of the difference tensor, are scalar multiples each other.

1. INTRODUCTION

Let M be an n-dimensional C^∞ manifold, and $F : M \to \mathbb{R}^{n+1}$ an immersion of class C^∞.
Consider the affine space $\mathbb{R}^{n+1}$ with its usual flat connection D and a volume form ω.

If we denote ξ the affine normal field and ∇ the induced connection (called normal connection) and S is the affine shape operator, we have the formula of Gauss

$$D_X Y = \nabla_X Y + h(X,Y) \qquad 1.1$$

and the formula of Weingarten

$$D_X\xi = -SX \qquad 1.2$$

The curvature tensor field R, which is of type (1, 3) is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \qquad 1.3$$

and the torsion tensor field T is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \qquad 1.4$$

If we denote $\widetilde{\nabla}$ the Levi-Civita connection for h, we can consider the difference tensor field K, which is of type (1,2), given by the diference between the normal and Levi-Civita connections, that is

$$K(X,Y) = \nabla_X Y - \widetilde{\nabla}_X Y$$

Because of ∇ and $\widetilde{\nabla}$ they are both torsion-free, we have $K(X,Y) = K(Y,X)$, and usually we write $K_X = \nabla_X - \widetilde{\nabla}_X$.
The covariant differentiation of K, with respect to Levi - Civita connection gives a tensor field $\widetilde{\nabla} K$, wuich is of (1, 3) type, and we ask ourselves if it feasible to have the equation $\widetilde{\nabla} K = \lambda R$ fulfilled (where λ is a scalar, possibily dependent on the dimension of M) for some hypersurfaces decomposable type.

2. Geomety of affine immersions

Let M be an n -dimensional manifold of class C^∞ and let $F : M \to \mathbb{R}^{n+1}$ be an affine immersion of class C^∞.
If ξ denotes the affine normal field and (∇, h, S) is the Blaschke structure on the hypersurface M, we have the fundamental equations

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY \qquad \text{Gauss equation} \qquad 2.1$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z) \qquad \text{Codazzi equation for } h \qquad 2.2$$

$$(\nabla_X S)Y = (\nabla_Y S)X \qquad \text{Codazzi equation for } S \qquad 2.3$$

$$h(SX,Y) = h(X,SY) \qquad \text{Ricci equation} \qquad 2.4$$

2.5 Definition. The Ricci tensor field Ric, which is of (0,2) type , is given by

$$\mathrm{Ric}(X,Y) = \mathrm{trace}\{Z \to R(Z,X)Y\}$$

2.6 Proposition. The Ricci tensor is given by $\mathrm{Ric}(X,Y) = \mathrm{Tr}(S)h(Y,Z) - h(SY,Z)$ and Ric=0 if, and only if $S = 0$.
Proof. [1]. Proposition 3.4, page 42.
Now, we define two more tensors, from ∇, $\widetilde{\nabla}$, and K.

$$\widetilde{L}(X,Y) = \mathrm{trace}\left\{Z \to \left(\widetilde{\nabla}_Z K\right)(X,Y)\right\} \qquad 2.7$$

and the analogous for the normal connection ∇

$$L(X,Y) = \mathrm{trace}\{Z \to (\nabla_Z K)(X,Y)\} \qquad 2.8$$

2.9. Proposition.

$$\widetilde{L}(X,Y) = \frac{1}{2}\mathrm{Tr}(S)h(X,Y) - \frac{n}{2}h(SX,Y) \qquad 2.10$$

$$\widetilde{L}(X,Y) - L(X,Y) = 2\mathrm{Tr}(K_X K_Y) \qquad 2.11$$

Proof. [1]. Propositions 9.4 y 9.9, pages 79 y 82 respectively.
From 2.10 we see immedialy that a hypersurface is an affine hypersphere $(S = \mu I)$ if, and only if $\widetilde{L} = 0$.

2.12. Proposition. If $\widetilde{\nabla} K = \lambda R$, then $\widetilde{L} = 0$, that is, the hypersurface is an affine hypersphere.

Proof. $\widetilde{\nabla} K = \lambda R$ implies $\left(\widetilde{\nabla}_Z K\right)(X, Y) = \lambda R(X, Y)Z$, then

$$\begin{aligned} \widetilde{L}(X, Y) &= \operatorname{trace}\left\{Z \to \left(\widetilde{\nabla}_Z K\right)(X, Y)\right\} \\ &= \operatorname{trace}\{Z \to \lambda R(X, Y)Z\} \\ &= \operatorname{trace}\{Z \to \lambda h(Y, Z)SX - \lambda h(X, Z)SY\} \\ &= \lambda h(Y, SX) - \lambda h(X, SY) \\ &= 0 \end{aligned}$$

by using (2.1) and (2.4).

2.13. Corollary. If $\widetilde{\nabla} K = \lambda R$, then $L(X, Y) = -2\operatorname{Tr}(K_X K_Y)$.
Proof. Is immediate, for (2.11).

3. Hypersurfaces Of Decomposable Type

We now add the hypothesis that the hypersurface can be expressed with respect to a suitable affine coordinate system by

$$F(t_1, \ldots, t_n) = (t_1, \ldots, t_n, f(t_1, \ldots, t_n))$$

with $(t_1, \ldots, t_n)$ varying in an open connected subset of R^n and f can be decomposed into a sum of n terms, each of them depending on only one of the independent variables $t_1, \ldots, t_n$, i.e.

$$f(t_1, \ldots, t_n) = f_1(t_1) + f_2(t_2) + \cdots + f_n(t_n)$$

The tensor field h of unimodular affine geometry, in terms of the graph function f, is

$$h = \sum_{i,j} h_{ij} dt_i \otimes dt_j \tag{3.1}$$

Where

$$h_{ij} = \delta_{ij} \phi^{-1} f_i''$$

and

$$\phi = \left(\prod_k |f_k''|\right)^{\frac{1}{n+2}}$$

If (h^{ij}) denotes the inverse matrix of (h_{ij}) the Christoffel symbols $\widetilde{\Gamma}^k_{ij}$ of Levi-Civita connection are given by

$$\widetilde{\Gamma}^k_{ij} = \frac{1}{2} \sum_m h^{km} (\partial_i h_{mj} + \partial_j h_{im} - \partial_m h_{ij}) \tag{3.2}$$

from which

$$\widetilde{\Gamma}^i_{ii} = \frac{n+1}{2(n+2)} \frac{f_i'''}{f_i''} \tag{3.3}$$

$$\widetilde{\Gamma}^i_{jj} = \frac{1}{2(n+2)} f_j'' \frac{f_i'''}{(f_i'')^2} \quad i \neq j \tag{3.4}$$

$$\widetilde{\Gamma}^i_{ij} = \frac{-1}{2(n+2)} \frac{f_i'''}{f_i''} \quad i \neq j \tag{3.5}$$

$$\widetilde{\Gamma}^k_{ij} = 0 \quad i \neq j \neq k \neq i \tag{3.6}$$

It can be seen, [2], that the Christoffel symbols $\widetilde{\Gamma}^k_{ij}$ of normal connection are given by

$$\Gamma^k_{ij} = h_{ij} \sum_m h^{km} \partial_m \log(\phi) \tag{3.7}$$

from which

$$\Gamma^i_{ii} = \frac{1}{n+2}\frac{f_i'''}{f_i''} \qquad 3.8$$

$$\Gamma^i_{jj} = \frac{1}{n+2} f_j'' \frac{f_i'''}{(f_i'')^2} \quad i \neq j \qquad 3.9$$

$$\Gamma^k_{ij} = 0 \quad \text{otherwise} \qquad 3.10$$

With the Christoffel symbols of both connections, we calculate the coefficients K^i_{jk} of the tensor difference K

$$K^i_{ii} = \frac{1-n}{2(n+2)}\frac{f_i'''}{f_i''} \qquad 3.11$$

$$K^i_{ij} = \frac{1}{2(n+2)}\frac{f_j'''}{f_j''} \quad i \neq j \qquad 3.12$$

$$K^i_{jj} = \frac{1}{2(n+2)} f_j'' \frac{f_i'''}{(f_i'')^2} \quad i \neq j \qquad 3.13$$

$$K^i_{jk} = 0 \quad \text{otherwise} \qquad 3.14$$

The coefficients S^i_j are given by

$$S^i_i = \frac{\phi}{(n+2)^2}((n+2)h_i - (2n+3)g_i) \qquad 3.15$$

$$S^i_j = \frac{\phi}{(n+2)^2}\frac{f_i'''f_j'''}{(f_i'')^2 f_j''} \quad i \neq j \qquad 3.16$$

Where $g_k = \frac{(f_k''')^2}{(f_k'')^3}$ and $h_k = \frac{f_k^{(4)}}{(f_k'')^2}$.

The derivatives g_k' and h_k' satisfies the following relations [3]

$$g_k' = \frac{f_k'''}{f_k''}(2h_k - 3g_k) \qquad 3.17$$

$$h_k' = \frac{1}{(f_k'')^2}\left(f_k^{(5)} - 2f_k'f_k'''h_k\right) \qquad 3.18$$

Using (3.15), (3.16) and the Gauss equation (2.19), we compute the component $R^i{}_{jkl}$ of the tensor R *as*

$$R^i_{jkl} = h_{kl}S^i_j - h_{jl}S^i_k \qquad 3.19$$

4. Covariant Differentiation of the Difference Tensor Multiple of Curvature Tensor.

This multiplicity is given by $\widetilde{\nabla} K = \lambda R,$ that in coordinates is equivalent to $K^i_{jk,l} = \lambda R^i_{jkl},$ where the components $K^i_{jk,l}$ are given by

$$K^i_{jk,l} = \partial_l K^i_{jk} + \sum_m \widetilde{\Gamma}^i_{lm} K^m_{jk} - \sum_m \widetilde{\Gamma}^m_{lj} K^i_{mk} - \sum_m \widetilde{\Gamma}^m_{lk} K^i_{jm} \qquad 4.1$$

We establish the following convention for the indices: for different symbols, correspond different values too, and they range from 1 to n. Hence, the equations that we consider are:

$$\frac{n-1}{4(n+2)}[(3n+5)g_i - 2(n+2)h_i] - \frac{3}{4(n+1)^2}\sum_{m\neq i} g_m = 0 \qquad 4.2$$

$$f_i'''f_j''' = 0 \qquad 4.3$$

$$2(n+2)h_i - (3n+5)g_i + (n+1)g_j - \sum_{m\neq i} g_m = 0 \qquad 4.4$$

$$(n-1)g_i + 2(n+2)h_j - (3n+5)g_j + \sum_{m\neq i} g_m = \lambda[4(n+2)h_i - 4(2n+3)g_i] \qquad 4.5$$

Remark: There are more equations, but they are omitted since they are redundant.

From (4.3) we see that at least n-1 functions must be of parabolic type. Denoting f_{k0} the remaining one and assuming that $f'''_{k_0} \neq 0$. Making $i = k_0$ in (4.2), (4.4) and (4.5) we have

$$2(n+2)h_{k_0} - (3n+5)g_{k_0} = 0 \qquad 4.6$$
$$(n-1)g_{k_0} = 4\lambda[(n+2)h_{k_0} - 4(2n+3)g_{k_0}]$$

and reordering

$$(3n+5)g_{k_0} - 2(n+2)h_{k_0} = 0 \qquad 4.7$$
$$(n-1+12\lambda+8\lambda n)g_{k_0} - 4\lambda(n+2)h_{k_0} = 0$$

and since $g_{k_0} \neq 0$, it must hold

$$\det\begin{bmatrix} 3n+5 & -2(n+2) \\ n-1+12\lambda+8\lambda n & -4\lambda(n+2) \end{bmatrix} = 0 \qquad 4.8$$

that is, $2(n+2)(2\lambda(n+1)+n-1) = 0$, from which $\lambda = \frac{1-n}{2(n+1)}$ and replacing in (4.7) we have

$$(3n+5)g_{k_0} - 2(n+2)h_{k_0} = 0$$

Hence,

$$h_{k_0} = \frac{3n+5}{2(n+2)}g_{k_0}$$

or equivalently

$$\frac{f^{(4)}_{k_0}}{(f''_{k_0})^2} = \left(\frac{3n+5}{2(n+2)}\right)\frac{(f'''_{k_0})^2}{(f''_{k_0})^3}$$

which solution, normalizing constants, is

$$f_{k_0}(t_{k_0}) = t_{k_0}^{-\frac{2}{n+1}}, \qquad t_{k_0} > 0 \qquad 4.9$$

Of course, the n functions there can not be of parabolic type, since in this case, the tensors K and R vanish.
We resume all this in the following theorem:

4.9 Theorem. Let M be a hypersurface of decomposable type with parametrization given by

$$(t_1, \ldots, t_n) \to (t_1, \ldots, t_n, f_1(t_1) + \cdots + f_n(t_n))$$

Then, the condition $\widetilde{\nabla} K = \lambda R$ hold, if $\lambda = \frac{1-n}{2(n+1)}$, $n-1$ of functions f_k are of parabolic type and the remaining one is given by $t \to t^{-\frac{2}{n+1}}$.

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