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ON FINITE DIMENSIONAL HILBERT SPACE FRAMES, DUAL AND NORMALIZED FRAMES AND PSEUDO- INVERSE OF THE FRAME OPERATOR.

L. Njagi[1*], B.M. Nzimbi[2] and S.K. Moindi[3]
[1]Department of Mathematics, Meru University of Science and Technology, P.O. Box 972- 60200, Meru.
[2,3]*School of Mathematics, University of Nairobi, Chiromo Campus, P. O. Box 30197-00100, Nairobi.*

Corresponding Author: -
E-mail: lnjagi@must.ac.ke

Abstract: -
In this research paper we do an introduction to Hilbert space frames. We also discuss various frames in the Hilbert space. A frame is a generalization of a basis. It is useful, for example, in signal processing. It also allows us to expand Hilbert space vectors in terms of a set of other vectors that satisfy a certain condition. This condition guarantees that any vector in the Hilbert space can be reconstructed in a numerically stable way from its frame coefficients. Our focus will be on frames in finite dimensional spaces.

Key words: *Hilbert space, frame, Dual frame, Psuedo-inverse, Normalized frames.*

1.0 INTRODUCTION

1.1 Hilbert Space

We introduce some basic definitions and facts about Hilbert space.

Definition 1. A Hilbert space is a complete, normed vector space H over the complex numbers $\mathbb{C}$, whose norm is induced by an inner product. The inner product is a function $\langle .,. \rangle : H \times H \rightarrow \mathbb{C}$, that satisfies:

(a) Linearity in the second argument: $\forall a, b \in \mathbb{C}$ and $\forall x, y, z \in H$,

$$\langle x, ay + bz\rangle = a\langle x, y\rangle + \langle x, z\rangle.$$

(b) Conjugate symmetry: $\forall x, y \in H$,

$$\langle x, y\rangle = \overline{\langle y, x\rangle}$$

where the overbar denotes complex conjugation.

(c) Positivity: $\forall x \neq 0 \in H$,

$$\langle x, x\rangle > 0.$$

The norm of H is induced by its inner product: $\forall x \in H$, $\|x\|^2 = \langle x, x\rangle$. The Hilbert space is required to be complete, which means that every sequence that is Cauchy with respect to this norm converges to a point in H.

The most common Hilbert spaces, and the only ones we shall be concerned with in this article, are the Euclidean spaces and the square-integrable function spaces.

Example 1. The Euclidean space $\mathbb{C}^n$ is a Hilbert space with an inner product defined by

$$(\langle x, y\rangle)^2 = \sum_{i=1}^{n} \bar{x}_i y_i.$$

The norm induced by this inner product is the standard Euclidean distance; for example, in $\mathbb{C}^2$ we have

$$\|x\| = \sqrt{x_1^2 + x_2^2}\ .$$

We can generalize these Euclidean spaces to infinite dimensions. Our vectors are then functions instead of n-tuples of numbers, and we must introduce the additional requirement that the functions be square-integrable to ensure that the inner product is well defined.

Example 2. For a measure space M and a measure μ, define $L^2(M, \mu)$ to be the set of measurable functions $f: M \rightarrow \mathbb{C}$ such that $\int |f|^2 d\mu < \infty$. This is a Hilbert space with the inner product

$$\langle f, g\rangle = \int \bar{f} g d\mu.$$

We shall take μ to be the Lebesgue measure when M is $\mathbb{R}$ or an interval on $\mathbb{R}$, and counting measure when $M = N$ or $M = \mathbb{Z}$.

In the first case, we use the notation $L^2([a, b])$, and the Hilbert space consists of square- integrable functions. In the second case, we use a lowercase *l* and write $l^2(N)$. Recall that integration with respect to counting measure is just summation, so the inner product on l^2 is

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i,$$

and l^2 is the space of square-summable sequences.

Hilbert spaces are "nicer" than general Banach spaces because of the additional structure induced by the inner product. The inner product allows us to define "angles" between vectors, and in particular, leads to the concept of orthogonality:

Definition 2. Two vectors x and y in a Hilbert space H are said to be **orthogonal** if $\langle x, y\rangle = 0$.

A set of vectors $\{x_i\}$ is said to be orthogonal if $\langle x_i, x_j\rangle = 0$ for $i \neq j$.

In the Euclidean spaces, the inner product is the standard dot product, and two vectors are orthogonal if their dot product is zero.

1.2 Linear Operators on Hilbert Space

Operator here means a linear map between two Hilbert spaces. We introduce some basic terminologies regarding operators.

Definition 3. For Hilbert spaces H_1 and H_2, mapping $T: H_1 \rightarrow H_2$, is called a **linear operator** if, for every $x, y \in H_1$ and for every $c_1, c_2 \in \mathbb{C}$, wehave

$$T(c_1x + c_2y) = c_1Tx + c_2\, Ty.$$

A linear operator is **bounded** if there exists a constant $k > 0$ such that $\|T x\| \leq k\, \|x\|$ for all nonzero $x \in H_1$. If T is a bounded operator, then we define the operator norm to be the norm induced by the two Hilbert space norms in the following way: $\|T\| = \inf\{k : \|Tx\| \leq k\ \|x\| \text{for all } x \neq 0\}$.

Every linear operator has an **adjoin**t, which is the unique operator T^* satisfying

$\langle Tx, y\rangle = \langle x, T^*y\rangle$, for all $x, y \in H1$.

An linear operator is an **injection** if T x = T y ⇒ x = y (that is, if T maps distinct elements in H1 to distinct elements in H2). A linear operator is a **surjection** if range(T) = H2. An operator that is both surjective and injective is called a **bijection.**

In the finite dimensional case, linear operators are just matrices; the linear operators from $\mathbb{C}^n$ to $\mathbb{C}^m$ are precisely the $\mathbb{C}^{m\times n}$ matrices in $\mathbb{C}^{m\times n}$. Infinite dimensional linear operators are the subject of functional analysis, and are much more difficult to classify in general. We will be working with a special type of linear operator called a "frame operator" whose norm is bounded above and below by two nonzero constants.

2.0 l^2 Representations of L^2 Functions

A common task in applied mathematics is to represent a function $f \in L^2$ in terms of some sequence of coefficients in l^2. For example, in signal processing applications we often represent an analog signal (an L^2 function) in terms of a sequence of coefficients. In theoretical considerations we may take these coefficients to be in l^2, but in practice we can only store finitely many coefficients. We hope to be able to choose a finite set of coefficients that capture most of the "information" in the original signal, in the sense that we can use the coefficients to reconstruct the original signal with a small L^2 error. The most common way to accomplish this task is to find a set of basis vectors $\{v_n\}$ for L^2, and use the inner products $\langle v_n, f\rangle$ as the l^2 coefficients representing a function $f \in L^2$.

Example 3. Consider the Hilbert space $L^2(0,1)$. The functions

$$\{e^{2\pi inx}, n \in \mathbb{Z}\}$$

form an orthonormal basis for this space, and we can represent any function $f \in L^2$ uniquely as a sequence in $l^2(\mathbb{Z})$ defined by

$$c_n = \langle e^{2\pi inx}, f\rangle, n \in \mathbb{Z}$$

These c_n are the "Fourier coefficients" of f. A function can be reconstructed from its Fourier coefficients using the inversion formula

$$f = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}$$

The map F that takes f to its sequence of Fourier coefficients c is unitary. This means that the problem of forming c given f and the inverse problem of forming f given c are both numerically stable, which is especially important in computational applications where we may only have an approximation to f or c. The fact that the operator F is bounded above and below, guarantees that, given a sufficiently good approximation for f, we can form an approximation of c.

It turns out that requiring the set $\{v_n\}$ to be a basis is overly restrictive for some applications. It is possible to form "stable representations" of arbitrary elements of H in terms of sets of vectors that are not necessarily linearly independent. The most general set of vectors that allows us to form stable representations of arbitrary vectors is called a "frame."

3.0 Hilbert Space Frames

A frame is a subset $\{\varphi_j\}$ of H that satisfies two very useful conditions:

(i) Every other vector in H can be written as a linear combination of the φ_j.

(ii) Every f in H can be represented as a sequence of "frame coefficients" in l^2, and each f can be reconstructed in a numerically stable way from its frame coefficients.

The frame coefficients of a function f are determined by applying the "frame operator" to f. Reconstruction of f from its frame coefficients is performed with a pseudoinverse.

Definition 4. Let $\{\varphi_j\}$ be a subset of H such that there exist $\propto, \beta > 0$ with

$$\propto \|f\|^2 \leq \sum (\varphi_j, f|)^2 \leq \beta \|f\|^2$$

for all $f \in H$. Then $\{\varphi_j\}$ is called a frame of H. The supremum of all $\propto$ and the infimum of all β that satisfy the above inequality are called the frame bounds.

The frame operator is the function F: $H \to l^2$ defined by

$(Ff)_n = \langle \varphi_n, f\rangle$.

By definition, the frame operator satisfies

$$\propto \|f\|^2 \leq \|Ff\|^2 \leq \beta\ \|f\|^2.$$

If $\propto = \beta$, then $\{\varphi_j\}$ is called a tight frame.

Some basic facts follow immediately from this definition.

(i) A frame must span the Hilbert space. Otherwise, F would have a non-trivial nullspace and there would be some f $\neq 0$ such that $\|Ff\|^2 = 0 < \propto \|f\|^2$.

(ii) The frame operator is an injection onto its range. If F f = F g, then by linearity F(f − g) = 0 and

$$\propto \|f - g\|^2 \leq 0 \leq \beta \|f - g\|^2 \Rightarrow \|f - g\| = 0 \Rightarrow f = g.$$

(iii) A frame does not have to be orthogonal, or even linearly independent.

Example 4. Let $H = \mathbb{C}^2$, and let $F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. The columns of $F^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ are a frame of $\mathbb{C}^n$, and F: $\mathbb{C}^2 \rightarrow \mathbb{C}^3$ is the associated frame operator. Its range in $\mathbb{C}^3$ is the span of its columns,

i.e., all vectors of the form $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$, and F is a bijection from $\mathbb{C}^2$ to this two-dimensional subspace of $\mathbb{C}^3$. Its frame bounds will be the squares of the singular values of F, which are the square roots of the eigenvalues of $F^*F = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Thus, $\propto = 1$ and $\beta = 3$.

The obvious question is how a vector $x \in \mathbb{C}^2$ can be reconstructed from its "frame coefficients" in $\mathbb{C}^3$. It turns out that there is another set of vectors in $\mathbb{C}^2$, called the dual frame, that is used to reconstruct x from its coefficients.

Before defining the dual frame, we compute the *adjoint* of the frame operator.

Proposition 1. Let {φj} ⊂ H be a frame and let F be the associated frame operator. Then the *adjoint* of F is the operator F^*: $l^2 \rightarrow$ H given by

$$F^*c = \sum c_j \varphi_j.$$

Proof. By definition, the adjoint satisfies

$$\langle Ff, c\rangle = \sum \langle \bar{\varphi}_j\ \bar{f}\rangle c_j$$

Using the conjugate symmetry and linearity of the inner product, this is

$$= \langle f, \sum c_j\varphi_j\rangle = \langle f, F^*c\rangle.$$

The result follows.

So, the adjoint of the frame operator takes a sequence c in l^2 to a linear combination of the frame vectors weighted by the coefficients cj.

3.1 The Dual Frame

Definition 5. Dual Frames

Let $\{\varphi_j\}$ be a frame in H. Then there is another frame $\{\tilde{\varphi}_j\} \subset$H, called the dual frame, given by:

$$\tilde{\varphi}_j = (F^*F)^{-1}\varphi_j.$$

It is instructive to look at the equivalent expression

$$\varphi_j = F^*F\tilde{\varphi}_j$$

$F\tilde{\varphi}_j$ is the l^2 sequence of "frame coefficients" of $\tilde{\varphi}_j$ in terms of the original frame; say $F\tilde{\varphi}_j = \tilde{c}$. The adjoint F^*, when applied to this sequence of coefficients, gives

$$F^*\tilde{c} = \sum \tilde{c}\varphi_i = \varphi_j$$

So, we have written each of the original frame vectors φj as a linear combination of the other φi, and the coefficients of this expansion are the inner products of the φi with the dual frame vector $\tilde{\varphi}_j$.

We will see below that we can expand any vector $f \in H$ as a linear combination of the φj, and the coefficients of this expansion will be given by the inner products of f with the dual frame vectors $\tilde{\varphi}_j$. On the other hand, we can write any vector f as a linear combination of the $\tilde{\varphi}_j$, and then the coefficients will be given by the inner products of f with the original frame. This is where the terminology "dual frame" comes from; reconstructing a vector from its frame coefficients and writing a vector as linear combination of frame vectors are in fact dual aspects of the same problem.

Proposition 2. If F is a frame operator (with frame bounds α and β), then F^*F is invertible; thus, the dual frame is well-defined.

Proof.

(i) Let $f \in H$ with $f \neq 0$. Then

$\langle F * Ff\rangle$, fi= $\langle Ff, Ff\rangle = \|Ff\|^2 \geq A \|f\|^2 > 0$.

It follows that $F{*}Ff \neq 0$, so F^*F is injective.

(ii) The range of F^*F is closed. Suppose g_n is some Cauchy sequence in the range of F^*F. That is,

$$\|g_n - g_m\| \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

and there is a sequence f_n such that $F^*Ff_n = g_n$ ∀n. For any n, m,

$$\langle F^*F(f_n - f_m), (f_n - f_m)\rangle = \|F(f_n - f_m\|^2 = \|g_n - g_m\|^2 \geq \alpha\|f_n - f_m\|^2,$$

since F is a frame with lower bound $\alpha > 0$. That means

$$\|f_n - f_m\|^2 \leq \frac{1}{A}\|g_n - g_m\|^2,$$

So $\|f_n - f_m\|^2 \to 0$ as n, m $\to \infty$ and f_n is a Cauchy sequence as well. Every Hilbert space is complete by definition, so f_n converges to some $f \in H$. F^*F is a bounded linear map: the norm of an operator and its adjoint are the same, so $\|F * F\| \leq \|F\|^2 = \beta$. It follows that F^*F is continuous, and

$F^*Ff_n = g_n \to F^*Ff \equiv g \in$ range(F^*F).

That is, any Cauchy sequence g_n in range (F^*F) converges to some element g in range($F*F$), and range(F^*F) is closed.

(iii). F^*F is a surjection, since

$\overline{range(F*F)}$= range(F^*F)=N(($F^*F)^*)^\perp$= N($F^*F)^\perp$ = $\{0\}^\perp$= H,

where we have used the fact that $F*F$ is self-adjoint.

Thus, F^*F is a bijection from H to itself and for every g $\in H$, there is a unique f $\in$H such that F^*Ff = g. The inverse is the unique operator satisfying $(F^*F)^{-1}g = f$.

We are now in a position to show that the dual frame is indeed a frame of H, and compute its frame operator and frame bounds.

Theorem 3. Suppose {φj} is a frame of a Hilbert space H, with associated frame operator $\tilde{F}$ and frame bounds $0 < \alpha \leq \beta < \infty$. Then the set $\{\tilde{\varphi}_j = (F^*F)^{-1}\varphi_j\}$ is another frame of H, with frame operator

$\tilde{F} = F(F^*F)^{-1}$,

Satisfying

$$\frac{1}{\beta}\|f\|^2 \leq \|\tilde{F}f\|^2 \leq \frac{1}{\alpha}\|f\|^2.$$

The set $\{\tilde{\varphi}_j\}$ is called the dual frame associated with the original frame.

Proof.

If $\{\tilde{\varphi}_j\}$ is to be a frame, then its frame operator is some $\tilde{F}$ satisfying :

$(\tilde{F}f)_j = \langle\tilde{\varphi}_j, f\rangle = (F^*F)^{-1}\tilde{\varphi}_j, f$. $(F^*F)^{-1}$ is the bounded inverse of a bounded self-adjoint operator, so it is self-adjoint, and

$$(\tilde{F}f)_j = \langle\varphi_j, (F^*F)^{-1}f\rangle.$$

By definition of the frame operator F, this is the j^th component of $F(F^*F)^{-1}$. Thus, the dual frame operator is given by $\tilde{F}=F(F^*F)^{-1}$.

To compute the frame bounds, we note that the inverse of a bounded self-adjoint operator is also self adjoint, so $\tilde{F}^* = (F^*F)^{-1}F^*$, and

$$\|\tilde{F}f\|^2 = \langle F^*Ff, f\rangle = \langle (F^*F)^{-1}F^*F(F^*F)^{-1}f, f\rangle$$

$$= \langle (F^*F)^{-1}f, f\rangle.$$

Let $g = (F^*F)^{-1}f$, so that

$$\|\tilde{F}f\|^2 = \langle g, F^*Fg\rangle = \|Fg\|^2.$$

Since F is a frame operator with bounds α and β,

$$\alpha\|g\|^2 \leq \|Fg\|^2 \leq \beta\|Fg\|^2.$$

Inserting $g = (F^*F)^{-1}f$ back into this inequality gives

$$A\|(F^*F^{-1}f\|^2 \leq \|f\|^2 \leq \|(F^*F)^{-1}f\|^2,$$

and rearranging, we have

$$\frac{1}{\beta}\|f\|^2 \leq \|(F^*F)^{-1}f\|^2 \leq \frac{1}{\alpha}\|f\|^2,$$

It follows that

$$\frac{1}{\beta}\|f\|^2 \leq \|\tilde{F}f\|^2 \leq \frac{1}{\alpha}\|f\|^2,$$

Thus, $\{\tilde{\varphi}\}$ is indeed a frame, with bounds $0 < \frac{1}{\beta} \leq \frac{1}{\alpha} < \infty$.

Example 5. Let us return to Example 4, and compute the dual frame. We have a frame $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ of $\mathbb{C}^2$ of $\mathbb{C}$. The frame operator is

$$F = \begin{pmatrix}1 & 0\\0 & 1\\1 & 1\end{pmatrix}$$

and the frame bounds are $\alpha = 1, \beta = 3$.
The dual frame operator is

$$\tilde{F} = F(F^*F)^{-1} = \begin{pmatrix}\frac{2}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & \frac{1}{3}\end{pmatrix}$$

so the dual frame is $\left\{\begin{pmatrix}2/3\\-1/3\end{pmatrix}, \begin{pmatrix}-1/3\\2/3\end{pmatrix}, \begin{pmatrix}1/3\\1/3\end{pmatrix}\right\}$. The bounds for the dual frame are $\tilde{A} = \frac{1}{\beta} = \frac{1}{3}$ and $\tilde{B} = \frac{1}{\alpha} = 1$.

Consider the vector $x = \begin{pmatrix}1\\2\end{pmatrix}$ Applying the operator F to it gives

$$Fx = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

These are the coefficients of the expansion of x in terms of the dual frame vectors; that is,

$$x = \begin{pmatrix}1\\2\end{pmatrix} = 1\begin{pmatrix}2/3\\-1/3\end{pmatrix} + 2\begin{pmatrix}-1/3\\2/3\end{pmatrix} + 3\begin{pmatrix}1/3\\1/3\end{pmatrix}$$

If we instead apply the dual frame operator to x, we find

$$\tilde{F}x = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

These are the coefficients of the expansion of x in terms of the original frame vectors:

$$X = \begin{pmatrix}1\\2\end{pmatrix} = 0\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} + 1\begin{pmatrix}1\\1\end{pmatrix}.$$

In general, the frame operator is not invertible, since it might not be surjective. However, in the example above, we were able to recover a vector x from its frame coefficients by writing it as a linear combination of the dual frame vectors; specifically,

$$\tilde{F}^*Fx = x$$

for any x, so the frame operator has a left inverse $\tilde{F}^*$ that inverts F on its range. This turns out to be true in general, and an analogous result allows us to recover a vector from its dual frame coefficients.

Theorem 4. Pseudo-Inverse of the Frame Operator

If F is a frame operator in a Hilbert space H and $\tilde{F}$ is the associated dual frame operator, then

$$\tilde{F}^*F = F^*\tilde{F} = I,$$

where I is the identity operator on H. That is, $\tilde{F}^*$ and F^* are left inverses for F and $\tilde{F}$ respectively.
Proof. The dual frame operator is given by

$$F = F(F^*F)^{-1}.$$

F^*F is a bounded, self-adjoint operator, so it is invertible and its inverse is self adjoint. Thus,

$$\tilde{F}^* = (F^*F)^{-1}F^*.$$

It follows that

$$\tilde{F}^*F = (F^*F)^{-1}(F^*F) = I.$$

Also,

$F^*\tilde{F} = F^*F(F^*F)^{-1} = I$. Thus, for any frame $\{\varphi j\}$ and its associated dual frame $\{\tilde{\varphi}\}$, we have for each $f \in H$

$$f = \sum\langle \emptyset_j, f\rangle \tilde{\emptyset}_j = \sum\langle \tilde{\emptyset}_j, f\rangle \emptyset_j.$$

When a frame is redundant (that is, F is not a surjection, or equivalently, the frame vectors are linearly dependent), $\tilde{F}^*$ is not a unique left inverse. We can add any arbitrary operator A that is zero on range(F) and still get a left inverse; if A: $l^2 \to H$ is a linear operator satisfying A (F f)= 0 for all $f \in H$, then clearly $(\tilde{F}^* + A)F f = \tilde{F}^*F f = f$ for any $f \in H$. The pseudo-inverse $\tilde{F}^*$ is chosen because it is zero on range(F)$^\perp$, and so it is "optimal" in the sense that is the left inverse with the smallest possible norm.

Theorem 5. "Optimality" of the Psuedo-Inverse

If F is a frame operator on H and $\tilde{F}$ is the dual frame operator, then $\tilde{F}^*$ is the left-inverse of F with minimum induced norm. That is, if $\tilde{F}^*$ F = TF = I, then

$$\|\tilde{F}\| \leq \|T\|.$$

Proof. First we show that range(F) is closed. Let $c_n = Ff_n$ be a Cauchy sequence in range(F); that is,

$$\|f_n - f_m\|^2 \to 0 \text{ as n, m} \to \infty.$$

From the frame inequality it follows that

$$\|F(f_n - f_m\|^2 = \|c_n - c_m\|^2 \geq \alpha \|f_n - f_m\|^2$$

so $\|f_n - f_m\|^2 \to 0$ as n, m$\to\infty$. Thus, f_n is a Cauchy sequence and it converges to some $f \in H$. F is bounded, so it is continuous, and F $f_n = c_n \to F f = c$ for some c ∈ range(F), and range(F) is closed.

Since range(F) is closed, we have

l^2 = range(F) ⊕ (range(F))$^\perp$.

Let c $\in l^2$ with c ≠ 0 and write c = c1 + c2 where c1 = F f ∈ range(F) and c2 ∈ (range(F))$^\perp$. Let T be an arbitrary left inverse of F. Then T and $\tilde{F}^*$ are equal on range(F), so

$$\frac{\|\tilde{F}^*c\|}{\|c\|} = \frac{\|\tilde{F}^*c_1\|}{\|c\|} = \frac{\|Tc_1\|}{\|c\|}$$

Since c1⊥c2, $\|c\|^2 = \|c_1\|^2 + \|c_2\|^2$ and $\|c\| \geq \|c\|$, so

$$\frac{\|\tilde{F}^*c\|}{\|c\|} = \frac{\|\tilde{F}^*c\|}{\|c\|} = \frac{\|Tc_1\|}{\|c\|} \leq \frac{\|Tc_1\|}{\|c\|} \leq sup\, \frac{\|Tc\|}{\|c\|}.$$

Thus,

$$\|\tilde{F}^*\| = \sup \frac{\|\tilde{F}^*c\|}{\|c\|} \leq \frac{\|Tc\|}{\|c\|} = \|T\|.$$

and the pseudo-inverse $\tilde{F}^*$ is the left inverse of F with minimum sup norm.

We have shown that the pseudo-inverse is the left inverse of minimum norm. If we know the lower frame bound α, then this norm is given by $\|\tilde{F}^*\| = \|\tilde{F}^*\| = \frac{1}{\sqrt{\alpha}}$. Having this bound on the pseudo-inverse is important for computational reasons; if $\frac{1}{\sqrt{\alpha}}$ is not too large, then a vector can be reconstructed from its frame coefficients in a numerically stable way. Say we have a vector $f \in H$ whose frame coefficients are given by F f = c. In practice, we will not have the exact frame coefficients, but some perturbed $\tilde{c}$ = c + δ where hopefully $\|\delta\| \ll 1$. Then the reconstructed vector will be $\tilde{f} = \tilde{F}^*\tilde{c} = f + \tilde{F}^*\delta$, and $\|\tilde{F} - f\| = \|\tilde{F}^*\delta\| \leq \frac{1}{\sqrt{A}} \|\delta\|$. Thus, small perterbations to the frame coefficients result in small perturbations of the reconstructed vector,

as long as $\frac{1}{\sqrt{\alpha}}$ is not too large.

3.2 Tight Frames

We have seen that in order to reconstruct a vector from its frame coefficients, we must have knowledge of the dual frame as well. For a special class of frames, this is no trouble, because the dual frame vectors are just constant multiples of the original frame vectors.

Theorem 6. The Dual Frame of a Tight Frame

A tight frame is a frame satisfying $\|Ff\|^2 = \alpha \|f\|^2$ for some $\alpha > 0$, and for every $f \in H$.

Let {φj} be a frame of H. Then {φj} is a tight frame with frame bound α if and only if the dual frame is given by $\{\tilde{\varphi}_j = \frac{1}{\alpha} \varphi_j\}$.

Proof.

Suppose {φj} is a tight frame. Then for any f ∈H,

$\|Ff\|^2 = \langle F * Ff, f\rangle = \alpha \|f\|^2 = \alpha \langle f, f\rangle$,

and thus

F∗F =AI

where I is the identity operator on H. It follows that $(F^*F)^{-1} = \frac{1}{\alpha} I$, and so $\tilde{\varphi}_j = \frac{1}{\alpha} \varphi_j$.

Conversely, suppose we know that the dual frame satisfies $\tilde{\varphi}_j = \frac{1}{\alpha}\varphi_j$. Then the associated frame operator satisfies $(\tilde{F} f)_j = \langle \tilde{\emptyset} j, f\rangle = \frac{1}{\alpha}(F f)_j$, so $F = \alpha \tilde{F}$ It follows from Theorem 4, then, that

$$F^*F = \alpha \tilde{F}^*F = \alpha I,$$

so for any $f \in H$,

$$\langle F * Ff, f\rangle = \|Ff\|^2 = \alpha\langle f, f\rangle = \alpha\|f\|^2$$

and $\{\varphi j\}$ is a tight frame with frame bound α.

Example 6. Any orthonormal basis is a tight frame with $\alpha = \beta = 1$.

Example 7. In $\mathbb{R}^2$, any set of 3 vectors that are equally distributed on the unit circle (meaning the angle between each of them is 120 degrees) will form a tight frame. For example, the set

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\cos\frac{2\pi}{3}\\ \sin\frac{2\pi}{3}\end{pmatrix}, \begin{pmatrix}\cos\frac{4\pi}{3}\\ \sin\frac{4\pi}{3}\end{pmatrix}\right\} = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1/2\\ \sqrt{3}/2\end{pmatrix}, \begin{pmatrix}-1/2\\ -\sqrt{3}/2\end{pmatrix}\right\}$$

is a tight frame. To see this explicitly, note that for any $v = \begin{pmatrix}x\\y\end{pmatrix}$ in $\mathbb{R}^2$, we have

$$\sum_{n=1}^{3} |\langle \emptyset_n, v\rangle|^2 = x^2 + \left(-\frac{x}{2} + \frac{\sqrt{3y}}{2}\right)^2 + \left(-\frac{x}{2} - \frac{\sqrt{3y}}{2}\right)^2$$

$$= x^2 + \frac{1}{4}x^2 + \frac{3}{4}y^2 - \frac{\sqrt{3}}{2}xy + \frac{1}{4}x^2 + \frac{3}{4}y^2 - \frac{\sqrt{3}}{2}xy$$

$$= \frac{3}{4}x^2 + \frac{3}{2}y^2 = \frac{3}{2}\|v\|^2.$$

3.3 Relationship between Frames and Bases

A frame is a generalization of a basis. Clearly, in finite dimensions a frame is a basis if and only if it is linearly independent; that is, a frame in $\mathbb{C}^n$ is a basis if and only if it consists of exactly n vectors.

Definition 6. Normalized Frames

Define a normalized frame to be a frame $\{\varphi j\}$ of a Hilbert space H such that $\|\varphi j\| = 1$ for all j . That is, a normalized frame consists of unit vectors.

Theorem 7. Conditions for a Normalized Frame to be an Orthonormal Basis

A normalized frame is an orthonormal basis if and only if $\alpha = \beta = 1$.

Proof. Suppose $\{vj\}j\in J$ is an orthonormal basis of a Hilbert space H. Then for any $u \in H$, we have the basis expansion $u = \sum_{j\in J}\langle v_j, u\rangle$ vj.

Since $\{vj\}$ is an orthonormal basis, $\langle v_j, v_i\rangle = \delta ij$ for all i, j $\in$ J. Thus,

$$\|u\|^2 = \langle u, u\rangle = \sum_{j\in J}\langle v_j, u\rangle \langle u, v_j\rangle = \sum_{j\in J}|\langle v_j, u\rangle|^2,$$

and vj is a normalized frame with $\alpha = \beta = 1$.

Now suppose $\{\varphi j\}j\in J$ is a normalized frame with $\alpha = \beta = 1$. Then for any $f \in H$,

$\|f\|^2 = \sum_{j\in J}|\langle \varphi_j, f\rangle|^2$.

Then in particular, for any i, $\|\varphi_i\|^2 = 1 = 1 + \sum_{j\neq i}|\langle \varphi_j + \varphi_i\rangle|^2$, which implies $|\langle \varphi_j, \varphi_i\rangle| = \delta ij$ so the frame vectors are orthonormal. To show that any vector can be expanded in terms of the frame vectors, consider the difference

$D = f - \sum_{j\in J}\langle \varphi_i, f\rangle \varphi j$.

This difference must satisfy

$\|d\|^2 = \sum_{j\in J}|\langle \varphi_j, d\rangle|^2$

$= \sum_{i\in J}(|\langle \varphi_i, f - \sum_{j\in J}\langle \varphi_j, f\rangle \varphi_j\rangle|)^2$

$= \sum_{i\in J}(|\langle \varphi_i, f\rangle - \sum_{j\in J}\langle \varphi_j, f\rangle\langle \varphi_i, \varphi_j\rangle|)^2$.

Since the φj are orthonormal, this is equal to

$\sum_{i\in J}(|\langle \varphi_i, f\rangle - \varphi_i, f|)^2 = 0$.

Thus, d=0 and $f = \sum_{j\in J}\langle \varphi_j, f\rangle \varphi_j$. It follows that $\{\varphi j\}$ constitutes an orthonormal basis.

If a frame is not normalized, then the result of Theorem 20 does not hold. We can show this by constructing a frame of $\mathbb{C}^2$ consisting of 3 vectors that has bounds $\alpha = \beta = 1$.

Example 8. To construct a frame of $\mathbb{C}^2$ consisting of 3 vectors that satisfies A = B = 1, we must find a matrix in $\mathbb{C}^{3\times 2}$ that has singular values $\sigma 1 = \sigma 2 = 1$. Such a matrix can be factored as USV, where U $\in \mathbb{C}^{3\times 3}$ and V $\in \mathbb{C}^{2\times 2}$ are unitary and S = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. For example, let

$$U = \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) & 1 \\ -\sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & 0 & \sin\left(\frac{\pi}{3}\right) \\ 0 & 1 & 0 \\ \sin\left(\frac{\pi}{3}\right) & 0 & \cos\left(\frac{\pi}{3}\right) \end{pmatrix}$$

And
V = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
This gives

$$USV = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & 0 \end{pmatrix},$$

So the set $\left\{ \begin{pmatrix} 1/4 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -\frac{3}{4} \\ 1/2 \end{pmatrix}, \begin{pmatrix} \sqrt{3/2} \\ 0 \end{pmatrix} \right\}$ is a tight frame of $\mathbb{C}^2$ with $\alpha = \beta = 1$. However, it is clearly not a basis. Theorem 7 is not violated because the frame vectors do not have unit length.

3.4 Frames in Finite Dimensions

We now consider frames in the finite dimensional Hilbert spaces $\mathbb{C}^n$. We give first a sufficient condition for a set of vectors to constitute a frame.

Lemma 8. Any finite, spanning set in $\mathbb{C}^n$ constitutes a frame.

Proof. Suppose $\{\varphi_j\}_{j=1}^{m} \subset \mathbb{C}^n$ and span($\{\varphi j\}$) $=\mathbb{C}^n$.

Define an operator F: $\mathbb{C}^n \to \mathbb{C}^m$ by $(Fx)j = \langle \varphi_j, x\rangle$. F can be written as a matrix F $\in \mathbb{C}^{m\times n}$. This matrix has rank n, since the span of its rows is all of $\mathbb{C}^n$ by assumption; it follows that all the singular values of F are nonzero. We denote the largest singular value by $\sigma 1$, and the smallest by σn. Then for any $x \in \mathbb{C}^n$,

$\sigma_n^2 \|x\|^2 \leq \|Fx\|^2 \sigma_1^2 \|x\|^2$.

Thus, F is a frame operator, $\{\varphi_j\}_{j=1}^{\infty}$ is a frame of $\mathbb{C}^n$, and the frame bounds are given by the squares of the smallest and largest singular values of F (i.e., the absolute values of the eigenvalues of F*F).

It is also possible to have an infinite frame in finite dimensions, as long as the length of the frame vectors goes to zero sufficiently fast.

Lemma 9. A countable spanning set $\{\varphi_j\}_{j=1}^{\infty}$ in $\mathbb{C}^n$ is a frame iff $\sum_{j=1}^{\infty} \|\varphi_j\|^2 < \infty$.

Proof. Let $\{\varphi_j\}_{j=1}^{\infty}$ be a set that spans $\mathbb{C}^n$.

Suppose iff $\sum_{j=1}^{\infty} \|\varphi_j\|^2 = \beta < \infty$. For any x $\in \mathbb{C}^n$, the Cauchy-Schwarz inequality gives

$$\|Fx\|^2 = \sum_{j=1}^{\infty} \left|\langle \varphi_j, x\rangle\right|^2 \leq = \|x\|^2 \sum_{j=1}^{\infty} \left\|\varphi_j\right\|^2 = \beta \|x\|^2,$$

so F is bounded above.

Since $\{\varphi j\}$ spans $\mathbb{C}^n$, we can choose a finite subset that also spans $\mathbb{C}^n$, say $\{\emptyset_i^1\}_{i=1}^{n}$n where $\varphi i' \in \{\varphi j\}$. By Lemma 17, this subset is a frame; say its lower frame bound is A. Then

$$\|Fx\|^2 = \sum_{j=1}^{\infty} \left|\langle \varphi_j, x\rangle\right|^2 \geq = \sum_{j=1}^{n} \left|\langle \varphi_j^1, x\rangle\right|^2 \geq A \|x\|^2,$$

and F is bounded below. It follows that F is a frame operator and the $\{\varphi j\}$ are a frame of $\mathbb{C}^n$.

4.0 Conclusions

We have introduced the concept of frames and gone through the basic definitions and important theorems. The real work, being in construction of frames that can be used in applications. References (1) and (2) provide constructions of wavelet frames and windowed Fourier frames, which have found great use in signal processing applications. The results we have presented about finite frames indicate that a normalized tight frame (FNTF) exhibits a great deal of symmetry, and that these FNTFs can be fully classified. The infinite dimensional analogous has not, I suppose, been fully explored; similar classification results for infinite dimensional normalized tight frames could give some insight into the concept of symmetry and “equidistribution” in the infinite dimensional setting.

5.0 Bibliography

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