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ON SOME FIXED-POINT RESULTS IN L^p SPACE

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Abstract:

The Banach fixed point theorem or contraction theorem of a complete metric space into itself .It state conditions sufficient for the existence and uniqueness of a fixed point

Keyword: *Fixed point, Lp space, metric space, complete metric space*

1. INTRODUCTION:

It is well Know that the Banach contraction principle is a fundamental result in the fixed point theory, which has been used and extended in many different direction, also there are severl generalizations of usual metric space and L^p space

Definition(1.1) (Metric space , Metric) : A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X) , that is a function defined on $X \times X$ such that for all $x, y, z \in X$ we have :
is real valued, finite and nonnegative. $(M1)$ d
$(x, y) = 0$ $\quad$ *if and only if* $x = y$.$(M2)$
$(x, y) = d(y, x)$ (Symmetry)$(M3)$
$(x, y) \le d(x, z) + d(z, y)$ (Triangle inequality) (1.1) $(M4)$

Sequence space l^∞ : as set X we take the set of all bounded sequences of complex numbers , that is every element of X is a complex sequence

$$(1.2)\ x = (\xi_1 , \xi_{,2}, \dots)\ \boldsymbol{briefly}\quad x = (\xi_j)$$

Such that for all $j = 1,2,\dots$ we have $|\xi_j| \le c_x$, where c_x is a real number which may depend on x , but does not depend on j . We choose the metric define by :

$$(x, y) = \sup_{j \in N} |\xi_j - \eta_j| \qquad (1.3)$$

Where $y = (\eta)_j \in X$ *and* $N = \{1,2, \dots \}$, and sup denotes the supremum " least upper bounded) .The metric space thus obtained is generally denoted by l^∞ is a sequence space because each element of X is a sequence .

2. Preliminaries : (I) l^p *ametric space*

Let ≥ 1 be a fixed real number . By definition , each element in the space $l^p$is a sequence $x = (\xi_j) = (\xi_1 , \xi_{,2}, \dots)$ of number such that $|\xi_1|^p + |\xi_2|^p + \cdots$ converges ;

Thus $\quad \sum_{j=1}^{\infty} |\xi_j|^p < \infty \qquad (2.1) \quad (\ge 1 , fixed)$

And the metric is defined by

$$d(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p\right)^{1/p} \quad (2.2)$$

Where $y = (\eta_j)$ *and* $\sum |\eta_j|^p < \infty$.

In the case $p = 2$, we have the famous Hilbert sequence space l^2 with metric defined by :

$$d(x, y) = \sqrt{\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2} \qquad (2.3)$$

This space was introduced and studied by D. Hilbert (1912) in connection with integral equations and is the earliest example of what is now called a Hilbert space.
We shall derive:

(a) **an auxiliary inequality** : Let $p > 1$ and defined q by

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad (2.4)$$

are then called conjugate exponents. This is a standard term. p *and* q
From (2.4) we have

$$1 = \frac{p + q}{pq}\ , pq = p + q\ , (p - 1)(q - 1) = 1 \quad (2.5)$$

Hence :

$$1/(p - 1) = q - 1\ , \text{so that}\ u = t^{p-1}\ \text{ implies }\ t = u^{q-1}$$

Let *and* β be any positive numbers. Since $\alpha\,\beta$ is the area of the rectangle, we thus obtain by integration the inequality

$$\alpha\,\beta \le \int_0^\alpha t^{p-1}\, dt + \int_0^\beta u^{q-1}\, du = \frac{\alpha^p}{p} + \frac{\beta^q}{q} (2.6)$$

Note that this inequality is trivially true if $\alpha = 0$ *or* $\beta = 0$

(b) **the Holder inequality from (a)** . Let $(\bar{\xi_j})$ *and* $(\bar{\eta_j})$ be such that

$$\sum|\bar{\xi_j}|^p = 1 , \quad \sum|\bar{\eta_j}|^q = 1 \qquad (2.7)$$

Setting $\alpha = |\bar{\xi_j}|$ *and* $\beta = |\bar{\eta}|$, we have from (2.6) the inequality

$$\sum|\bar{\xi}_j\bar{\eta}_j| \leq \frac{1}{p} + \frac{1}{q} = 1 \qquad (2.8)$$

We now take any nonzero $x = (\xi_j) \in l^p$ *and* $y = (\eta_j) \in l^q$ and set

$$(2.9)\bar{\xi}_j = \frac{\xi_j}{\left(\Sigma|\bar{\xi}_k|^p\right)^{1/p}} \quad , \bar{\eta}_j = \frac{\eta_j}{\left(\Sigma|\bar{\eta}_j|^q\right)^{1/q}}$$

Then (2.7) is satisfied , so that we may apply (2.8) . Substituting (2.9) into (2.8) and multiplying the resulting inequality by the product of the denominators in (9)

, we arrive at the **Holder inequality** for sums

$$\sum_{j=1}^{\infty}|\xi_j\eta_j| \leq \left(\sum_{k=1}^{\infty}|\xi_k|^p\right)^{1/p}\left(\sum_{m=1}^{\infty}|\eta_m|^q\right)^{1/q} \qquad (2.10)$$

Where >1 and $\frac{1}{p} + \frac{1}{q} = 1$ .This inequality was given by O. Holder

If $= 2$, *then* $q = 2$ and (2.10) yields the Cauchy –Schwarz inequality for sums

$$\sum_{j=1}^{\infty}|\xi_j\eta_j| \leq \sqrt{\sum_{k=1}^{\infty}|\xi_k|^2}\sqrt{\sum_{m=1}^{\infty}|\eta_m|^2} \qquad (2.11)$$

It is too early to say much about this case $p = q = 2$ in which p equals its conjugate q , but we want to make at least

(c) the Minkowski inequality from (b) : for sums

$$\left(\sum_{j=1}^{\infty}|\xi_j + \eta_j|\right)^{1/P} \leq \left(\sum_{k=1}^{\infty}|\xi_k|^P\right)^{1/P}\left(\sum_{m=1}^{\infty}|\eta_m|^2\right)^{1/P} \qquad (2.12)$$

Where $x = (\xi_j) \in l$ and $y = (\eta_j) \in l^p$ and $p \geq 1$

For $= 1$the inequality follows readily from the formulas we shall write $\xi_j + \eta_j = \omega_j$. The triangle inequality for numbers gives

$$|\omega_j|^p = |\xi_j + \eta_j||\omega_j|^{p-1} \leq (|\xi_j| + |\eta_j|)|\omega_j|^{p-1}$$

Summing over j from 1 to any fixed n, we obtain

$$\sum|\omega_j|^p \leq |\xi_j||\omega_j|^{p-1} + \sum|\eta_j||\omega_j|^{p-1} \qquad (2.13)$$

To the first sum on the right, we apply the Holder inequality, finding

$$\sum|\xi_j||\omega_j|^{p-1} \leq \left[\sum|\xi_k|^p\right]^{1/p} + \left[\sum(|\omega_m|^{p-1})^q\right]^{1/q}$$

On the right we simply have: $(p-1)\,q = p$ because $pq = p + q$

Treating the last sum in (2.13) in a similar fashion, we obtain

$$\sum|\eta_j||\omega_j|^{p-1} \leq \left[\sum|\eta_k|^p\right]^{1/p} + \left[\sum(|\omega_m|^{p-1})^q\right]^{1/q}$$

Together,

$$\sum|\omega_j|^p \leq \left\{\left[\sum|\xi_k|\right]^{1/p} + \left[\sum|\eta_k|^p\right]^{1/p}\right\}\left(\sum|\omega_j|^p\right)^{1/p}$$

Dividing by the last factor on the right and noting that $1 - 1/q = 1/p$

We obtain (2.12) with n instead of ∞. We now let $n \to \infty$ hence the series on the left also converges, and (2.12) is proved.

(d) the triangle inequality (M4) from (c): From (2.12) it follows that for x *and* y *in* l the series in (2.2) converges. (2.12) also yields the triangle inequality.

In fact , taking any $x, y, z \in l^p$, writing $z = (\xi_j)$ and using the triangle inequality for numbers and then (12) , we obtain

$$d\left(\sum|\xi_j - \eta_j|^p\right)^{1/p}(x,y) = \left(\sum[\xi_j - \zeta_j| + |\zeta_j - \eta_j]^p\right)^{1/p}$$

$$\leq \left(\sum|\xi_j - \zeta_j|^p\right)^{1/p} + \left(\sum|\xi_j - \eta_j|^p\right)^{1/p}$$

$$= d(x,z) + d(z,y)$$

$$\Longrightarrow l^p \text{ is ametric space} \quad \#$$

: here p is fixed and $1 \leq p < +\infty$ **(*II*) *Completeness of* l^p**

Let (x_n) be any Cauchy sequence in the space l , where $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \dots)$
Then for every $\epsilon > 0$ there is an N such that for all $m, n > N$,

$$d(x_m, x_n) = \left(\sum_{j=1}^{\infty}\left|\xi_j^{(m)} - \xi_{j1}^{(n)}\right|^p\right)^{1/p} < \epsilon \quad (2.14)$$

It follows that for every $j = 1,2, \dots$ we have

$$|\xi^{(m)} - \xi_{j1}^{(n)}| < \epsilon \qquad (m, n > N) \quad (2.15)$$

We choose a fixed j .From (2.15) we see that $(\xi_j^{(1)}, \xi_j^{(2m)}, \dots)$ isa Cauchy sequence of numbers .say $\xi_j^{(m)} \to \xi_j$ *as* $m \to \infty$ .Using these limits we define $x = (\xi_1, \xi_2, \dots)$ and show that $x \in l^p$ and $x_m \to x$ .From (2.14) we have for all

$$m, n > \ , \quad \sum_{J=1}^{K}|\xi_j^{(m)} - \xi_{j1}^{(n)}| < \epsilon^P \qquad (k = 1,2, \dots)$$

Letting $n \to \infty$, we obtain for $m > N$, $\sum_{J=1}^{\infty}|\xi_j^{(m)} - \xi_{j1}^{(n)}| < \epsilon^P$ (2.16)

This show that $_m - x = \xi_j^{(m)} - \xi_{j1}^{(n)} \in l^p$ .Since $x_m \in l^p$it follows by means of the Minkowski inquality (2.12) that $x = x_m + (x - x_m) \in l^p$.
Furthermore , the series in (2.16) represents $[d(x_m, x_n)]^p$, so that (2.16) implies that $x_m \to x$ .Since (x_m) was an arbitrary Cauchy sequence inl^p .this proves
compleness of l^p , where $1 \leq p \leq +\infty$.

(III) **Space l^p i s separable:** The space $\boldsymbol{lp}$ *with* $1 \leq p \leq +\infty$ is separable. Proof: Let M be the set of all sequences y of the form

$$y = (\eta_1, \eta_2, \dots, \eta_n, 0,0, \dots)$$

is any positive integer and the η_j *'s* are rational M is countable .We *where n* show that M is dense in $\boldsymbol{l^p}$. Let $= (\boldsymbol{\xi_j}) \in l^p$ be arbitrary . Then for every $\epsilon > 0$
there is an n (depending on ϵ) *such that* $\sum_{j=n+1}^{\infty}|\xi_j|^p \leq \frac{\epsilon^P}{2}$

have the remainder of a converging series . Since the *because on the left w* rational are dense in $\boldsymbol{R}$, for each there is a rational η_J close to it

$$\sum_{j=1}^{n}|\xi_j - \eta_J|^p \leq \frac{\epsilon^P}{2}$$

It follows that :

$$[\boldsymbol{d}(\boldsymbol{x},\boldsymbol{y})]^p = \sum_{j=1}^{n}|\xi_j - \eta_J|^p + \leq \sum_{j=n+1}^{\infty}|\xi_j|^p \leq \frac{\epsilon^P}{2}$$

We thus have $(\boldsymbol{x},\boldsymbol{y}) < \epsilon$ and see that M is dense in l^p

(IV) **Theorem (2.1) (Fischer – Riesz)** : l^p **is a Banch space** for any $1 \leq p \leq \infty$
Proof : We distinguish the cases $p = \infty$ *and* $1 \leq p < \infty$

Case (1) : $p = \infty$. Let (f_n) be a Cauchy sequence is L^∞ , given an integer $k \geq 1$ there is an integer N_k such that $\| f_m - f_n\|_\infty \leq \frac{1}{k}$ for $m, n \geq N_k$. Hence there is a null set E_k such that

$$| f_m(x) - f_n(x)| \leq \frac{1}{k} \quad \forall x \in \Omega/E_k \quad \forall m,n \geq N_k \quad (2.17)$$

Then we let $E = \cup_k E_k$ so that E is a null set and we see that for all The sequence $f_n(x)$ is Cauchy $(in\ \mathbb{R})$ .Thus $f_n(x) \to f(x)\ x \in \Omega \backslash E$ For all $x \in \Omega \backslash E$ .Passing to the limit in (2.17)as $m \to \infty$ we obtain

$$| f_m(x) - f_n(x)| \leq \frac{1}{k} \quad for\ all\ x \in \Omega \backslash E \quad \forall n \geq N_k$$

We conclude that $f \in L^\infty$ and $\|f - f_n\|_\infty \leq \frac{1}{k} \ \forall n \geq N_k$

Therefore $f_n \to f$ *in* L^∞.

Case (2) : $1 \le p < \infty$. Let (f_n) be a Cauchy sequence in L^p .In order to conclude , it suffices to show that a subsequence converges in L^p
We extract a subsequence (f_{nk}) such that

$$|f_{n_{k+1}} - f_{n_k}|_p \le \frac{1}{2^k} \quad \forall k \ge 1$$

We claim that f_{nk} converges in ^{p} .In order to simplify the notation we write f_k instead of f_{nk} , so that we have

$$|f_{k+1} - f_k|_p \le \frac{1}{2^k} \quad \forall k \ge 1 \qquad (2.18)$$

Let

$$g_n(x) = \sum_{k=1}^{n} |f_{k+1}(x) - f_k(x)| \ \textit{so that} \ \| g_n\|_p \le 1$$

As a consequence of the monotone convergence theorem $g(x)$ tends to a finite limit , say $g(x)$,*a. e on* Ω , *with* $g \in L^P$ .On the other hand m for $m \ge n \ge 2$ we have

$$|f_m(x) - f_n(x)| \le |f_m(x) - f_{m-1}(x)| + \cdots + |f_{n+1}(x) - f_n(x)|$$

$$\le g(x) - g_{n-1}(x)$$

It follows that a.e Ω, $f(x)$ is Cauchy and converges to a finite limit , say $f(x)$.
We have a.e on Ω , $|f(x) - f_n(x)| \le g(x) \quad for\ n \ge 2 \quad (2.19)$

And in particular $f \in L$. Finally , we conclude by dominated convergence that

$$, since\ |f_n(x) - f(x)|^p \to 0\ a.e. \|f_n - f\|_p \to 0$$

$$and\ also |f_n - f|^p \le g^P \in L^1 \ \#$$

3. Main Results: Banach Fixed Point Theorem:

A fixed Point : of a mapping $T: X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself , that is $Tx = x$, the image Tx coincides with x .

Definition (3.1) (Contraction): let $X = (X, d)$ be a metric space.
A mapping $T: X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$:

$$(Tx, Ty) \le \alpha\, d\,(x, y)\ (\alpha < 1) \qquad (3.1)$$

Geometrically this mean that any points x *and* have images thay are closer together than those points x *and* y , more precisely the ratio $d(\,Tx, Ty\,)/d(x, y)$ does not exceed a constant α which is strictly less than 1.

Banach Fixed Point Theorem (3.2) (Contraction Theorem) :
Consider a metric space$X = (X, d)$, where $X \ne \emptyset$ .Suppose that X is complete and let$T: X \to X$ be a contraction on X .Then T has precisely one fixed point .
Proof : We construct a sequence (x_n) and show that it is Cauchy , so that it converges in the complete space X , and then we prove that its limit x is a fixed point of T and T has no further fixed points . This is the idea of the proof .
We choose any $_0 \in X$ and define the " iteravtive sequence " (x_n) by x_0 , $x_1 =$

$$Tx_{0\,,2} = Tx_1 = T^2x_0\,, \ldots , x_n = T^n \qquad (3.2)$$

Clearly , this the sequence of the images of x_0 under repeated application of T
.We show that (x_n) is Cauchy . By (3.1) and (3.2)

$$d(x_{m+1}, x_m) = d(Tx_{m+1}, Tx_{m-1}) \le \alpha\, d(x_m, x_{m-1})$$

$$= \alpha\,(Tx_{m-1}, Tx_{m-2}) \le \alpha^2 d(x_{m-1}, x_{m-2}) \ldots \le \alpha^M d(x_1, x_0)$$

Hence by the triangle inequality and the formula for the sum of a geometric progression we obtain for $n > m$

$$d(x_m, x_n) = d(x_m, Tx_{m+1}) \le d(x_{m+1}, 2) + \cdots + d(x_{n-1}, x_n)$$

$$\le (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1})\, d(x_{0,} x_1)$$

$$= \alpha^m \ \frac{1 - \alpha^{n-m}}{1 - \alpha}\ d\,(x_{0,} x_1)$$

Since $0 < \alpha < 1$, in the numerator we have $1 - \alpha^{n-m} < 1$. Consequently

$$d(x_m, x_n) \leq \frac{\alpha^m}{1-\alpha} \; d\,(x_{0,}x_1) \quad (\mathbf{n} > m) \qquad (\mathbf{3.3})$$

On the right $0 < \alpha < 1$ and $d(x_{0,}x_1)$ is fixed , so that we can make the right hand side as small as we pleas by taking m sufficiently large

This prove that (x_m) is Cauchy. Since X is complete (x_m) converges, say

$x_m \to x$. We show that limit x is a fixed point of the mapping T

From the triangle inequality and (3.1) we have

$$(x, Tx) \leq d(x, x_m) + d(x_m, Tx) \leq d(x, x_m) + \alpha\, d(x_{m-1}, x)$$

And can make the sum in the second line smaller than any preassigned $\in > 0$ because $x_m \to x$ .We conclude that $d(x, Tx) = 0$ so that $x = Tx$ by (M2) .This shows that x is affixed point of T. is the only fixed point of T because form $Tx = x$ and $T\tilde{x} = \tilde{x}$ obtain by (3.1) : $d(x, \tilde{x}) = d(Tx, T\tilde{x}) \leq \alpha\, d(x, \tilde{x})$

which implies $d(x, \tilde{x})$=0 since $\alpha < 1$ .Hence $x = \tilde{x}$ by (M2) and the theorem is prove #

corollary (3.3) (Iteration , error bounds) : Under the conditions of Theorem (3.2) the iterative sequence (3.2) with arbitrary $x_{0,} \in X$ converges to the unique fixed point x of T .Error estimates are the prior estimate

$$d(x_m, xx_n) \leq \frac{\alpha^m}{1-\alpha} \; d\,(x_{0,}x_1) \qquad \textit{(3.4)}$$

and the posterior estimate

$$d(x_m, xx_n) \leq \frac{\alpha^m}{1-\alpha} \; d\,(x_{m-1,}x_m) \qquad \textit{(3.5)}$$

Proof : the first statement is obvious form the previous proof. Inequality (3.4) follows from (3.3) by letting $n \longrightarrow \infty$ .We derive (3.5)

Taking $m = 1$ and writing y_0 for x_0 and y_1 for x_1 , we have from (3.4)

$$d(y_1, x) \leq \frac{\alpha^m}{1-\alpha} \; d\,(y_0, y_1)$$

Setting $_0 = x_{m-1}$, we have $y_1 = Ty_0 = x_m$, and obtain (3.5)

Theorem (3.4) : (Contraction on a ball)

Let T be a mapping of a complete metric space $X = (X, d)$ into itself. Suppose is a contraction on a closed ball is T satisfies (3.1) for all $x, y \in Y$. Moreover, assume $Y = \{x|\, d(x, x_0) \leq r\}$, $that$ that

$$(x_0, Tx_0)(1 - \alpha)r \qquad (3.6)$$

Then the iterative sequence (3.2) converges to an $x \in Y$ .This x is a fixed point of $T$and is the only fixed point of T in Y .

Proof : We merely have to show that all $x_m's$ as well as x lie in Y. We put $m = 0$ in (3.4) , change n to m and use (3.5) to get

$$d(x_0, x_m) \leq \frac{1}{1-\alpha}\, d(x_0, x_1) < r$$

Hence all $x_m's$ are in Y. also $x \in Y$ since $x_m \to x$ and Y is closed. The assertion of the theorem now follows from the proof of Banach's theorem

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