EXISTENCE AND UNIQUENESS OF SOLUTION OF SOME BOUNDARY VALUE PROBLEMS FOR DELAY DIFFERENTIAL EQUATIONS

Dr. Nedal. Hassan E. ELjaneid[1]*

[1]Department of Mathematics, Faculty of Sciences, University of Tabuk, Kingdom of Saudi Arabia

*Corresponding Author: -

Abstract: -

We presented as study to establish the existence and uniqueness of solution some three-point boundary value problems for second order differential equations.

Keyword: *Boundary Value Problems, Delay Differential Equations*

1. INTRODUCTION:

This paper is concerned with some existence and   of solutions for some boundary value problem for Delay differential equations:

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t), x(t - \tau_0), \dot{x}(t - \tau))$$
$$x(t) = \dot{x}(t) = 0 , t < 0 , x(\eta) = \alpha x(1), x(0) = 0 , 0 < \eta < 1 \quad (1.1)$$

It should be noted that the differential equations have penetrated all branches of contemporary science. The goal is how to use a creative method to study the existence and uniqueness of problem of three-dimension boundary values from the second level of the late nonlinear differential equations. The composition   method will be used when dealing with the second problem.

2. Preliminaries

2.1 : Initial Conditions and Boundary conditions:

In mathematics, in the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions

For example, if the independent variable is time over the domain [0,1],   a boundary value problem would specify values for $y(t)$ both $t = 0$ and $t = 1$, whereas an initial value problem would specify a value of $y(t)$ and $y(t)$ at time $t = 0$

2.2 : The contraction mapping:

A contraction in a metric space is a mapping $f: X \rightarrow X$ such that there exists $0 < c < 1$ such that for all $x, y \in X, x \neq y$, it holds that $d(f(x), f(y)) \leq cd(x, y)$. Indeed, the Banach fixed-point theorem shows that such mappings on complete metric spaces always have a unique fixed point.

2.3 : The fixed point:

Fixed points of functions in the complex plane   commonly lead to beautiful   fractal structures. For example, the plots above color the value of the fixed point (left figures) and the number of iterations to reach a fixed point (right figures) for cosine (top) and sine (bottom). Newton's method, which essentially involves a fixed point computation in order to find roots, leads to similar fractals in an analogous way. Points of an autonomous system of ordinary differential equations at which $\dfrac{dx_1}{dt} = f_1(x_1, x_2.., x_n) = 0 , ..., \dfrac{dx_n}{dt} = f_n(x_1, x_2.., x_n) = 0$ are known as fixed points.

3. Main Results:

In this part we will prove a single following solution of boundary value problem

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t), x(t - \tau_0), \dot{x}(t - \tau))$$
$$x(t) = \dot{x}(t) = 0, t < 0, x(\eta) = \alpha x(1), x(0) = 0 , 0 < \eta < 1 \quad (3.1)$$

Suppose the operator $f(t, x, y, z, w)$ is define and continues in field

$$D = \{[0,1] \times Q_1 \times Q_2 \times Q_1 \times Q_2\}$$
$$Q_1 = \{Z: |Z|_1 \leq \mathscr{R}_1\} \quad \& \quad Q_1 = \{w: |w|_1 \leq \mathscr{R}_2\}$$

also $\tau_0(t), \tau_1(t)$ are positive continues functions in interval [0,1]
The equations $t - \tau(t) = 0$ have a finite solutions λi,

$$\ddot{x}(t) - k^2 x(t) = f(t, x(t), \dot{x}(t), x(t - \tau_0), \dot{x}(t - \tau) - k^2 x(t))$$
$$x(t) = \dot{x}(t) = 0, t < 0, x(0) = 0 \; x(\eta) = \alpha x(1), 0 < \eta < 1 \quad (3.2)$$

So, the boundary problem (3.2) t can be converted by using the method parameters into the equation the integrally equivalent, to find a general solution

$$\ddot{x}(t) - k^2 x(t) = 0 \Rightarrow x_h = c_1 \cosh kx + c_2 \sinh kx \quad (3.3)$$

Suppose the general solution

$$x_p(t) = u(t) \cosh kx + v(t) \sinh kx \quad (3.4)$$

Satisfying the following conditions:

$$\dot{u}(t) \cosh kt + \dot{v}(t) \sinh kt = 0$$
$$\dot{u}(t) = f(t, x(t), \dot{x}(t), x(t - \tau_0), \dot{x}(t - \tau) - k^2 x(t))$$

By solving the last two equations we get

$$u(t) = \frac{-1}{k} \int_0^t \sinh ks \ (f(t,x(t),\dot{x}(t),x(t-\tau_0),\dot{x}(t-\tau)-k^2x(t)) \ ds$$

$$v(t) = \frac{1}{k} \int_0^t \cos ks \begin{bmatrix} f(t,x(t),\dot{x}(t),x(t-\tau_0),\dot{x}(t-\tau)) \\ -k^2x(t)) \qquad (3.5) \end{bmatrix} ds$$

Substitute in equations (3.4) to (3.5)

$$x(t) = \frac{1}{k\,(\sinh k\eta - \alpha \sinh k)} (\sinh k\eta - \alpha \sinh k)$$

$$\int_0^t \sinh k \ (t-s) \ k^2 x(s) - f(s,x(s),x(s),x(s-\tau_0(s),x(\ s-\tau_1(s)ds$$

$$+ + \ \alpha \int_0^1 \sinh kt \ \sin k(1-s)\,[k^2x(s) - f(s,x(s),x(s),x(s-\tau_0(s),x(\ s-\tau_1(s)]$$

$$- \int_0^n \sinh kt \ \sin k(\eta - s)\,[k^2x(s) - f(s,x(s),x(s),x(s-\tau_0(s),x(\ x(s$$

$$- \tau_0(s)] \ ds$$

$$x(t) = \frac{1}{(\sinh k\eta - \alpha \sinh k)} (\sinh k\eta - \alpha \sinh k)$$

$$\int_0^t \cosh(t-s) \ k^2x(s) - f(s,x(s),x(s),x(s-\tau_0(s),x(\ s-\tau_1(s)ds$$

$$+ \alpha \int_0^1 \cosh kt \ \sin k(1-s)\,[k^2x(s) - f(s,x(s),x(s),x(s-\tau_0(s), \ x(s-\tau_0(s)]\ d\!s$$

$$- \int_0^n \cosh kt \ \sin k(\eta - s)\,[k^2x(s) - f(s,x(s),x(s),x(s-\tau_0(s), \qquad (3.7)$$

Theorem of existence and uniqueness of problem (3.1):

In boundary value problem (3.1), suppose that :

(1) $\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial w_2}$ continuous for every value $t \in [0,1]$, $\forall z_1, w_1, z_2, w_2$

(2) $L_1, L_2, L_3 \ \ and \ L_4$ positive constants

$$0 \le \frac{\partial f}{\partial z_1} \le L_1, 0 \le \frac{\partial f}{\partial w_1} \le L_2, 0 \le \frac{\partial f}{\partial z_2} \le L_3, \le \frac{\partial f}{\partial w_2} \le L_4, \forall t \in [0,1]$$

$$and \ \ \forall z_1, w_1, z_2, w_2$$

(3) $\frac{M}{k^2} \left| \dfrac{\sinh k\,(1+\eta) - \sinh k\eta - sinhk}{sinhk - \alpha \sinh k} \right| \le R_1$

$\frac{M}{k} \left| \dfrac{\cosh k(1+\eta) + \alpha(1 - \cosh k) - cosk}{sinhk - \alpha \sinh k} \right| \le R_2 \qquad (3.8)$

There is only one solution of problem (1) on condition $\| A_K \| < 1$

$$A_K = \begin{bmatrix} \dfrac{|k^2 - L_1 - L_3|}{k^2}\left(\dfrac{a}{c}\right) & \dfrac{|L_2 + L_4|}{k^2}\left(\dfrac{a}{c}\right) \\ \dfrac{|k^2 - L_1 - L_3|}{k}\left(\dfrac{a}{c}\right) & \dfrac{|L_2 + L_4|}{k}\left(\dfrac{a}{c}\right) \end{bmatrix} \qquad (3.9$$

$$a = |\ \sinh k\,(1-\eta) - \sinh k\eta - sinhk|$$

$$b = |\cosh k\,(1+\eta) + \alpha(1 - \cosh k) - cosk|$$

$$c = |sinhk - \alpha \sinh k\,|$$

(4) Furthermore it $\forall k : k \ge \max\left(\frac{L_1 + L_3}{2}, \frac{L_2 + L_4}{2}\right)$

$$x_0(t) = 0, \dot{x}_0(t) = 0$$

$$x_{m+1}(t) = \frac{1}{k(\sinh k\eta - \alpha \sinh k)} \{(\sinh k\eta - \alpha \sinh k)$$

$$\int_0^t \sinh k(t-s)\,[k^2 x_m(s) - f(s,x_m(s),x_m(s),x_m(s-\tau_0(s),x_m(s - \tau_1(s)]ds \qquad + \alpha \int_0^1 \sinh kt \ \sin k(1-s)\,[k^2 x_m(s) \ -$$

$$f(s,x_m(s),x_m(s),x_m(s-\tau_0(s),x_m((s-\tau_1(s))]$$

$$- \int_0^n \sinh kt \ \sin k(\eta - s)\,[k^2 x_m(s) \ - f(s,x_m(s),x_m(s),x_m(s-\tau_0(s),x_m((s-\tau_1(s))] \ ds$$

$$\int_0^t \sinh k(t-s)\left[k^2 x_m(s) - f(s, x_m(s), x_m(s), x_m(s-\tau_0(s)), x_m(s - \tau_1(s)]ds \right. + \alpha \int_0^1 \sinh kt \, \sin k(1-s)\left[k^2 x_m(s) - f(s, x_m(s), x_m(s), x_m(s-\tau_0(s)), x_m((s-\tau_1(s)))\right]$$

$$-\int_0^n \sinh kt \, \sin k(\eta - s)\left[k^2 x_m(s) - f(s, x_m(s), x_m(s), x_m(s-\tau_0(s)), x_m((s-\tau_1(s)))\right]ds$$

$$x_{m+1}(t) = \frac{1}{(\sinh k\eta - \alpha \sinh k)}\left\{(\sinh k\eta - \alpha \sinh k)\right.$$

$$\int_0^t \cosh k(t-s)\left[k^2 x_m - (s)f(s, x(s), x_m(s), x_m(s-\tau_0(s)), x_m(s-\tau_1(s)))\right]ds$$

$$+\alpha \int_0^1 \cos h\, k(1-s)\left[k^2 x_m(s) - f(s, x_m(s), x_m(s), x_m(s-\tau_0(s)), x_m((s-\tau_1(s)))\right]$$

$$-\int_0^n \cosh kt \, \sin k(\eta - s)\left[k^2 x_m(s) - f(s, x_m(s), x_m(s), x_m(s-\tau_0(s))\right]ds \quad (3.10)$$

proof: Define the space C [0,1], the opretors U, V, G

Suppose the vector space functions $\quad c: y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} : x(t), \dot{x}(t) \quad$ are

continuous functions $\quad \|y(t)\|_c = max \begin{bmatrix} \underset{0 \le t \le 1}{max} \|x(t)\| \\ \underset{0 \le t \le 1}{max} \|\dot{x}(t)\| \end{bmatrix}$

let operator c :

$$G(y(t)) = \left\{ \frac{1}{k(\sinh k\eta - \alpha \sinh k)} \quad \{\sinh k\eta - \alpha \sinh k\right)$$

$$\int_0^t \sinh k \,(t-s)\, k^2 x(s) - f(s, x(s), x(s), x(s-\tau_0(s)), x(s-\tau_1(s))ds$$

$$++\,\alpha \int_0^1 \sinh kt \, \sin k(1-s)\left[k^2 x(s) - f(s, x(s), x(s), x(s-\tau_0(s)), x(s-\tau_1(s))\right]$$

$$-\int_0^n \sinh kt \, \sin k(\eta - s)\left[k^2 x(s) - f(s, x(s), x(s), \dot{x}(s-\tau_0(s)), \dot{x}(s\right.$$
$$\left.-\tau_1(s)))ds\right\}$$

$$\left\{\frac{1}{(\sinh k\eta - \alpha \sinh k)}\{(\sinh k\eta - \alpha \sinh k)\right.$$

$$\int_0^t \cosh\, k(t-s)\, f(s, x_m(s), x_m(s), x_m(s-\tau_0(s)), x_m(s-\tau_1(s))]ds \quad + \alpha \int_0^1 \cosh kt \, \sin(1-s)[f - k^2 x_m(s)]ds$$

$$-\int_0^n \cosh kt \, \sin(\eta - s)[f - k^2 x_m(s)]\, ds \quad (3.10)$$

Suppose $\quad y(t) \in \check{S} : \check{S} = \{y(t) \in c: \|y(t)\|_c \le max(R_1, R_2)\}$

Use condition (3.3):

$$U(y(t)) \le \frac{M}{K^2} \frac{\sinh k(1+\eta) - \sinh k\eta - \sinh k)}{(\sinh k\eta - \alpha \sinh k)} \le R_1$$

and

$$V(y(t)) \le \frac{M}{k} \left| \frac{\cosh k\,(1+\eta) + \alpha(1 - \cosh k) - \cos k}{\sinh k - \alpha \sinh k} \right| \le R_2$$

$G(y(t)) \in \check{S}$

In other words, we find that the operator $G(y(t))$ is transports closed and limited set and convexity S for the same set.

To prove that we only need to prove that two functions $x_m(t), x_m(t)$

Representing the Cauchy sequence

(i) $l_{m+1}(t) = x_{m+1}(t) - x_m(t)$

(ii) $h_{m+1}(t) = \dot{x}_{m+1}(t) - \dot{x}_m(t)$

(3.11) (iii) $\| l_{m+1}(t)\| = max_{0\leq t\leq 1}| l_{m+1}(t)|$, $m = 0,1,2, ..$

(iv) $\| h_{m+1}(t)\| = max_{0\leq t\leq 1}| h_{m+1}(t)|$, $m = 0,1,2, ..$

$$l_{m+1}(t) = \frac{1}{k(\sinh k\eta - \alpha \sinh k)}\{\{(\sinh k\eta - \alpha \sinh k)$$

$$\left\{\int_0^t \sinh k(t-s)\left[\left(k^2 - \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2}\right)l_m(t) - \left(\frac{\partial f}{\partial w_1} - \frac{\partial f}{\partial w_2},\right)h_m(t)\right\} ds\right.$$

$$+ \alpha \int_0^1 \sinh kt\ \sin k(1-s)\left[k^2 - \left(k^2 - \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2}\right)l_m(t) - \left(\frac{\partial f}{\partial w_1} + \left(\frac{\partial f}{\partial w_2}\right)h_m(t)\right] ds\right.$$

$$- \int_0^n \sinh kt\ \sin k(\eta - s)\left(k^2 - \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2}\right)l_m(t) - \left(\frac{\partial f}{\partial w_1} + \left(\frac{\partial f}{\partial w_2}\right)h_m(t)\right] ds \quad (3.12)$$

$$l_{m+1}(t) = \frac{1}{k(\sinh k\eta - \alpha \sinh k)}\{\{(\sinh k\eta - \alpha \sinh k)$$

$$\left\{\int_0^t \cosh k(t-s)\left[\left(k^2 - \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2}\right)l_m(t) - \left(\frac{\partial f}{\partial w_1} - \frac{\partial f}{\partial w_2},\right)h_m(t)\right\} ds\right.$$

$$+ \alpha \int_0^1 \sinh kt\ \sin k(1-s)\left[k^2 - \left(k^2 - \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2}\right)l_m(t) - \left(\frac{\partial f}{\partial w_1} + \left(\frac{\partial f}{\partial w_2}\right)h_m(t)\right] ds\right.$$

$$- \int_0^n \cosh kt\ \sin k(\eta - s)\left(k^2 - \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2}\right)l_m(t) - \left(\frac{\partial f}{\partial w_1} + \left(\frac{\partial f}{\partial w_2}\right)h_m(t)\right] ds \quad (3.13)$$

and whereas $k^2 \geq max\left(\frac{L_1+L_3}{2}, \frac{L_3+L_4}{2}\right)$:

$$\begin{bmatrix} \|l_{m+1}(t)\| \\ \|h_{m+1}(t)\| \end{bmatrix} \leq \begin{bmatrix} \dfrac{|k^2 - L_1 - L_3|}{k^2}\left(\dfrac{a}{c}\right) & \dfrac{|L_3 + L_4|}{k^2}\left(\dfrac{a}{c}\right) \\ \dfrac{|k^2 - L_1 - L_3|}{k^2}\left(\dfrac{a}{c}\right) & \dfrac{|L_3 + L_4|}{k^2}\left(\dfrac{a}{c}\right) \end{bmatrix}\begin{bmatrix} \|l_m(t)\| \\ \|h_m(t)\| \end{bmatrix} \quad (3.1.$$

Equation (3.14) can be written

$$Z_{m+1} \leq \|A_k\| Z_m , m = 1,2,..$$

$$or : Z_{m+1} \leq \|A^m{}_k\| Z_1 , \|A^m{}_k\| \leq \|A_k\|^m$$

$$Z_{m+1} \leq \|A^m{}_k\| Z_1 \quad (3.15)$$

If $\|A\|\leq 1 \Rightarrow x_m(t)$, $\dot{x}_m(t)$ is Cauchy sequences in continues function space in interval [0,1] according to the standard defined so exist continuous limit $x(t)$, $\dot{x}(t) \to x_m(t)$, $\dot{x}_m(t)$, and to be two processes limit and integral are commutative so limit of function satisfies two integral equations

To prove uniqueness, suppose that $v(t)$, $u(t)$ are two solutions thus, they satisfy the equations (3.6), (3.7) and with the same analysis that leads to

the equation (3.15): $\|u - v\| \leq A \|u - v\|$, if $\|A\|\leq 1$. It must be in accordance with the contraction mapping $\|u - v\| = 0 \Rightarrow u - v = 0 \Rightarrow u = v$

This is the end of proving the existence and uniqueness of solution to boundary problem (3.1)

Example (3.2): Let boundary problem:

$$x(t) = 0.1\left[\frac{x(t)}{4}\sin t + \dot{x}(t-0.5), t \in [0,1]\right]$$

$$x(t) = \dot{x}(t) = 0 , t < 0\ x(o) = 0.2\ and\ x(0,5) = x(1)$$

$$f(t, y_1, z_1, y_2, z_2) = 0.1\left(\frac{y_1}{4}\sin t + z_2\right)\ and\ let\ R_1 = 0.5, R_2 = 2$$

$$M = 0.210518387 , L_1 = 0.022 , L_2 = L_3 = 0 , L_4 = 0.1$$

$$\begin{pmatrix} 0.0055 & 0.025 \\ 0.0055 & 0.025 \end{pmatrix} \Rightarrow \|A\| < 1$$

$$x_{m+1}(t) = 0.2 - 0.1 \int_0^t S \left[\frac{x_m(s)}{4} \sin s + \dot{x}_m(s - 0.5) \right] ds$$

$$+ \int_0^{0.5} t \left[\frac{x_m(s)}{4} \sin s + \dot{x}_m(s - 0.5) \right] ds$$

$$+ 2t \int_{0.5}^1 (1 - s) \left[\frac{x_m(s)}{4} \sin s + \dot{x}_m(s - 0.5) \right] ds$$

$$\dot{x}_{m+1}(t) = -0.1 \int_t^{0.5} \left[\frac{x_m(s)}{4} \sin s + \dot{x}_m(s - 0.5) \right] ds$$

$$+ 2 \int_{0.5}^1 (1 - s) \left[\frac{x_m(s)}{4} \sin s + \dot{x}_m(s - 0.5) \right] ds$$

$$x_0(t) = 0.24 \, , x_0(t) = 1$$

$$y_1(t) := 0.2 - 0.00200000000 \, t^3 - 0.05000000000 \, t^2 -$$

$$0.003000000000 \, t \, (0.25 - 1. t^2) - 0.1t \, (0.5 - 1. t) - 0.0260000000$$

$$z_1(t) := -0.07675000000 + 0.003000000000 \, t^2 + 0.1t$$

$$z_1(t) \equiv x_1(t) \implies y_1 \equiv x_1 \quad \#$$

Acknowledgment

References:
[1]. Andreson,J.and Thompson ,H.B.Existence of multiple solutions for second order boundary value problems ,.J.Differential Equations ,Vol 166 (2000) .
[2]. Liu, B.Positive solutions of a nonlinear three point boundary value problem , Appl .Math .Comput ,Vol 132 (2002)
[3]. Agarwal, R.P,Philos .Ch.G. and Tsamatos ,P.CH., Global solutions of a singular initial value problem to second order nonlinear delay differential equations , Math compute .Modelling ,Vol 43. (2006).
[4]. Yog .H,Liu, Study of singular boundary value problems for second order impulsive differential equations ,J.Math.Anal .Apple ,Vol 331 (2007)
[5]. Agarwal, R.P.O'Regan, Dm and Yan,B,positive solutions of a singular initial value problem to second three point boundary value problems , J.differential equations , Vol 116. (2008).
[6]. Lin, X,Du,Z, and Liu ,W, Uniqueness and existence results for a third order nonlinear multi point boundary value problem ,Apple .Math .Comput , Vol 205 (2008) .
[7]. Kumar,M.and Singh ,N,A collection of computational techniques for solving singular boundary value problems, Adances in Engineering Software , Vol 40 (2009) .