DOI: https://doi.org/10.53555/nnms.v4i3.543 **Publication URL: http://nnpub.org/index.php/MS/article/view/543**

MATRIX BASED ON THE SECOND DERIVATIVE OF INFINITE CONVERGENT GEOMETRIC SERIES

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Abstract: -

The infinite Geometric Series is a series of the form $\sum_{k=0}^{\infty} ax^k$ *, where a is a constant. The geometric power series* $\sum_{k=0}^{\infty} ax^k$ *converges for* $|x|<1$ *and is equal to* $\frac{a}{1-x}$. *The Second Derivative of*

$$\sum_{k=0}^{\infty} ax^k \text{ is } \sum_{k=2}^{\infty} ak(k-1)x^{k-2} = \sum_{k=0}^{\infty} a(k+2)(k+1)x^k$$

Let t be sequence in (0,1) that converges to 1. The matrix based on second derivative of convergent infinite geometric series defined as $a_{nk} = \frac{1}{2}(k+2)(k+1)(1-t_n)^3 t_n^{\,k}$. *We denote this matrix by* S_t *and name it the matrix associated second derivative of geometric series.* S_t *is a sequence to sequence mapping. When a matrix* S_t *is applied to a sequence x, we get a new sequence* $S_t x$ *whose nth term is given by:*

$$(S_t x)_n = \frac{1}{2}(1-t_n)^3 \sum_{k=o}^{\infty} (k+2)(k+1)t_n^{\,k} x_k$$

The sequence $S_t x$ *is called the* S_t*-transform of the sequence x.*
The purpose of this research is to investigate the effect of applying S_t *to convergent sequences, bounded sequences, divergent sequences, and absolutely convergent sequences. We considering and answer the following interesting main research questions.*

I.

Research Questions.
(1) What is the domain of t for which S $_t$ maps convergent sequence into convergent sequence?
(2) What is the domain of t for which the S $_t$ maps absolutely convergent sequence into absolutely convergent sequence?
(3) Does S $_t$ maps unbounded sequence to convergent sequence?
(4) Does S $_t$ maps divergent sequence to convergent sequence?
(5) How is the strength of the S $_t$ comparing to the identity matrix?

Notations and Background Materials
w= {the set of all complex sequences}
c= {the set of all convergent complex sequences}

$$c(A) = \{y: Ay \in c\}$$

$$l = \{y: \sum_{k=0}^{\infty} |y_k| < \infty\}$$

$$l(A) = \{y: Ay \in l\}$$

Definition 1: A matrix A is an x-y matrix if the image Au of u under the transformation A is in Y wherever u is in x.

Regular Matrix
A matrix is regular if $lim_{n\to\infty} Z_n$= a⇒ $lim_{n\to\infty}(AX)_n$=a. That is a sequence Z is convergent to A⇒ the A-transform of Z also converses to a.

The Sliverman-Toeplitz Rule
We state the following famous Sliverman-Toeplitz Rule as Proposition I without proof and apply it.

Proposition I: A matrix A =($a_{n,k}$) is regular if and only if

(i) $\lim_{n\to\infty} a_{n,k} = 0$ for each k= 0,1,...,

(ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1,$ and

(iii) $\sup_n \{ \sum_{k=0}^{\infty} |a_{n;k}|\} \le M < \infty$ for some $M < 0.$

The Main Results
Theorem 1: The S $_t$ matrix is a regular matrix for all t.
Proof: We use proposition 1, to prove the theorem. Note that

(1) $\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} \frac{1}{2}(k+2)(k+1)(1-t_n)^3 t_n{}^k \quad =0$

(2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \lim_{n\to\infty} \frac{1}{2}\sum_{k=0}^{\infty}(k+2)(k+1)t_n{}^k(1-t_n)^3 =$

$\lim_{n\leftarrow\infty} \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty}(k+2)(k+1)t_n{}^k = \frac{(1-t_n)^3}{(1-t_{n)})^3} = 1$ and

(3) $Sup_n \sum_{k=0}^{\infty} a_{n,k} = 1$

Hence by Proposition I, the matrix S_t is **a regular matrix**. Thus, the matrix S $_t$ maps all convergent sequences into convergent sequences and we can say that the matrix S_t a c-c matrix.

Remark 1: The S $_t$ matrix maps a bounded sequence into a convergent sequence as shown by the following example. This shows that the S $_t$ matrix is stronger than the identity matrix or c(S $_t$) is larger than c.

Example 1: Consider the bounded sequence given by x_k =$(-1)^k$

$$\text{Then } (S_t x)_n = \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty}(k+2)(k+1)(t_n)^k(-1)^k$$

$$= \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty}(k+2)(k+1)(-t^n)^k$$

Remark 2: The S_t matrix maps also a divergent sequence x into a convergent sequence as shown by the following example.

Example 2: Consider the unbounded sequence given by x defined by $x_k = (-1)^k (k+3)$. Note that

$$(S_t x)_n = \frac{1}{2}\sum_{k=0}^{\infty}(1-t_n)^3 t_n^k (-1)^k x_k (k+2)(k+1)$$

$$= \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty} t_n^k (-1)^k)(k+3)(k+2)(k+1)$$

$$= (1-t_n)^3 \sum_{k=0}^{\infty} (-t_n)^k (k+3)(k+2)(k+1)$$

$$= \frac{3(1-t_n)^3}{(1+t_n)^4}$$

Now, $\lim_{n\to\infty} (S_t x)_n = \lim_{n\to\infty} \frac{(1-t_n)^3}{(1+t_n)^4} = 0$

Hence $S_t x \in c.$

Knopp-Lorentz Thorem

The Matrix A is an $\ell - \ell$ matrix if and only if there exists a number M>0 such that for every k,

$$\sum_{n=0}^{\infty} |a_{nk}| \le M.$$

Theorem 2: S_t is $\ell - \ell \Leftrightarrow (1-t)^3 \in \ell$

Lemma 1:

$$S_t \text{ is } \ell - \ell \text{ matrix} \Rightarrow (1-t)^3 \in \ell .$$

Proof: We use the Knopp-Lorentz Rule.

$$S_t \text{ is } \ell - \ell \Rightarrow \sum_{n=0}^{\infty} |(1-t_n)^3 t_n^k| \le M$$

$$\Rightarrow \sum_{n=0}^{\infty} |(1-t_n)^3| \le M \quad \text{(for k=0)}$$

$$\Rightarrow (1-t)^3 \in \ell$$

Lemma 2:

$$(1-t)^3 \in \ell \Rightarrow S_t \text{ is an } \ell - \ell \text{ matrix}$$

Proof: We use the Knopp-Lorentz Rule

$$\sum_{n=0}^{\infty} |a_{nk}| \le \sum_{n=0}^{\infty} |(1-t_n)^3 t_n^k|$$

$$\le \sum_{n=0}^{\infty} (1-t_n)^3 \le M \text{ for some M>0 as } (1-t)^3 \in \ell .$$

Now Theorem 2 follows by Lemmas 1&2.

Corollary 1. $\arcsin(1-t)^2 \in l \Leftrightarrow S_t$ is an l-l matrix.

Proof: The corollary easily follows using Theorem 2 and the following basic inequality.

$$(1-t)^3 \le \arcsin(1-t)^3 \le \frac{(1-t)^3}{\sqrt{1-(1-t)^3}}.$$

Theorem 3 $\frac{-1}{\ln(1-t_n)} \in l \Rightarrow \mathbf{S}_t$ is an l-l matrix.

Proof. Note that:

$$(1-t_n)^3 \le (1\text{-}t_n) := \left(\sum_{k=0}^{\infty} t_n^{\,k}\right)^{-1}$$

$$\le \left(\sum_{k=0}^{\infty} \frac{1}{k+1} t_n^{\,k}\right)^{-1}$$

$$= \left(\sum_{k=0}^{\infty} t_n^{\,k}\left(\int_0^1 V^k dV\right)\right)^{-1}$$

$$= \left(\sum_{k=0}^{\infty}\left(\int_0^1 t_n^{\,k} V^k dV\right)\right)^{-1}$$

$$= \left(\int_0^1 dV\left(\sum_{k=0}^{\infty} (t_n V)^k\right)\right)^{-1}$$

The Interchanging of the Integral and summation is legitimate as the power series $\sum_{k=0}^{\infty} \left(Vt_n\right)^k$ converges absolutely and uniformly for $0 \le Vt_n \le 1$. Hence we have,

$$(1-t_n)^3 \le 1\text{-}t_n \le \left(\int_0^1 \frac{dV}{1-Vt_n}\right)^{-1}$$

$$= \left(\frac{-1}{t_n}(\ln(1-t_n)\right)^{-1}$$

$$\le \frac{-1}{\ln(1-t_n)}$$

The hypothesis that $\frac{-1}{\ln(1-t)} \in l \Rightarrow (1\text{-}t)^3 \in l$ and hence by Theorem 2, S_t is l-l.

Remark 3. An l-l S_t matrix maps a bounded sequence into l as shown by the following example. This shows that the S_t matrix is stronger than the identity matrix in the l-l setting or $l(S_t)$ is larger than l.

Example 3.
Assume the S_t matrix is l-l and consider the bounded sequence given by $x_k = (-1)^k$

$$\text{Then } (S_t x)_n = \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty} (k+1)(k+2)(t_n)^k(-1)^k$$

$$= \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty} (k+2)(k+1)(-t^n)^k$$

$$= (1-t_n)^3 \frac{1}{(1+t_n)^3}$$

$$\le (1-t_n)^3$$

Now the S_t matrix is $l\text{-}l \Rightarrow (1\text{-}t)^3 \in l$, by Theorem 2, and hence $S_t x \in l.$

Remark 4: An l-l S_t matrix maps unbounded sequence into l as shown by the following example.

Example 4: Assume S_t is an l-l matrix and consider the unbounded sequence given by $x_k = (-1)^k (k+3)$. Note that

$$(S_t x)_n = = \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty} t_n^k (-1)^k (k+3)(k+2)(k+1)$$
$$= \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty} (-t_n)^k (k+3)(k+2)(k+1)$$
$$= \frac{(1-t_n)^3}{(1+t_n)^4}$$
$$\leq (1-t_n)^3$$

Important Dedication: This paper is dedicated to my Great wife Mrs. Aster Debebe for her extraordinary support on my research activities. I am able to be a good mathematician because of her. Thank you Aster.

Acknowledgements:
Special thanks to Dawit Getchaw, Eneye Negassa, Samea Mulatu, Abyssinia Mulatu, Sara Worku, Lela Mikre, Benni Fekadu and Akalewolde Wegayhu.

Reference:

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